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Klein-Bottle *Tonnetze*

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ABSTRACT: Departing from the toroidal *Tonnetz* of neo-Riemannian theory, we construct a generalized Klein-bottle *Tonnetz*. Further, we examine associated transformational graphs and analytical contexts, using operators from the cyclic T group, the dihedral T/I group, and a generalized quaternion T/M subgroup. In the T/I example, corresponding regions within a *Tonnetz* are related by various Klumpenhouwer network isographies. Finally, we consider relationships among entire Klein-bottle *Tonnetze*, and place them into recursive supernetworks.

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Bibliography

Appendix

[1] INTRODUCTION

[1.1] Our point of departure is the “Table of Tonal Relations,” or *Tonnetz*, used by Hugo Riemann (1902, 479) and other nineteenth-century German music theorists to model relationships among triads and keys.[1] A rendering of this table appears in Figure 1.

Figure 1. Riemann’s (1902) Table of Tonal Relations

				gisis	disis	aisis				
			ais	eis	his	fisis	cisis			
		h	fis	cis	gis	dis	ais	eis		
	c	g	d	a	e	h	fis	cis	gis	
des	as	es	b	f	c	g	d	a	e	h
	fes	ces	ges	des	as	es	b	f	c	
		asas	eses	heses	fes	ces	ges	des		
			feses	ceses	geses	asas	eses			

More recently, neo-Riemannian theorists have resurrected the *Tonnetz* as a network on which to illustrate certain transformational relationships.[2] It has been modularized to accommodate pitch-class space, essentially forming a grid on the surface of a torus, and has been further generalized using techniques from graph theory and abstract algebra.[3] In the present study, we consider the implications of a particular algebraic relation on a class of related *Tonnetze*, using various groups of GIS-intervals.[4] The fundamental regions underlying their graphs describe Klein bottles;[5] therefore, we will refer to these constructs as “Klein-bottle *Tonnetze*.”[6]

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[1] For a discussion of the history and development of various *Tonnetze*, see Cohn (1997, 7–10; and 1998, 171–73) and Mooney (1996).

[2] Such transformational relationships originate with REL, PAR, and LT in Lewin (1982 and 1987).

[3] Hyer (1989) was the first to substitute pitch-class integers for letter names. Cohn (1997, Note 14) cites Lubin (1974) with having first observed the toroidal geometry of an enharmonically equivalent *Tonnetz*. Examples of generalized *Tonnetze* appear in Cohn (1997, 10–21) and Gollin (1998, 202–03).

[4] For a brief discussion of Generalized Interval Systems (GIS’s), see Lewin (1987, 26–30).

[5] The Klein bottle, named after mathematician Felix Klein, is a type of manifold, or unbounded surface. However, unlike a sphere or a torus, a Klein bottle has no inside or outside.

[6] In the case when the group of GIS-intervals is commutative, the fundamental region actually degenerates into a torus, not a Klein bottle. See Note 39.

[1.2] Klein-bottle *Tonnetze* that incorporate both T_n and I_n operators are related to Perle cycles,[7] and we may use both types of networks to study Klumpenhouwer networks.[8] This relationship is an anti-isomorphism;[9] hence, a T/I Klein-bottle *Tonnetz*[10] and the Perle cycle to which it is anti-isomorphic have the same algebraic structure. However, they have very different surface features. Corresponding regions within Perle cycles demonstrate strong Klumpenhouwer-network isographies,[11] while the pitch-class contents of these segments belong to varying set-classes.[12] In contrast, corresponding regions in T/I Klein-bottle *Tonnetze* display weak isographies, but the segments are members of the same set-class.[13]

[1.3] Lewin (2002, 197) points out that Perle cycles are useful in Klumpenhouwer-network analysis, as they provide a method for modeling strong isographies. Their weakness, then, resides in the fact that they cannot be used to demonstrate recursive structures. Furthermore, neither the Perle-cyclic nor the recursive analytical method addresses specifically the set-class content of the pitch-class sets they interpret. Whereas this information may be of secondary importance in a transformational analysis, it is often musically salient, especially in passages with restricted set-class content. Klein-bottle *Tonnetze* are capable of modeling recursion, particularly with regard to musical contexts in which the set-class content is circumscribed. Indeed, such modeling is useful, as the musical literature contains many instances of limited-set-class passages that suggest Klumpenhouwer-network interpretations.

[1.4] Example 1a shows one such excerpt, the opening canon of no. 8, "Nacht," from Arnold Schoenberg's *Pierrot Lunaire*, op. 21. Example 1b demonstrates how the passage may be interpreted as a cycle of imbricated 3-3[014] trichords. Figure 2 presents a network of the passage, wherein these trichords appear as adjacent triangular subnetworks, labeled g1 through g9. As we will see later, this network is a Klein-bottle *Tonnetz*. Whereas the interpretation of 3-3[014] trichords elsewhere in the piece might suggest the exclusive use of T_n arrows,[14] I_n arrows are appropriate here if we wish to show some degree of recursion. The inversional relation between adjacent trichords suggests their inclusion.

[7] George Perle develops the concept of Perle cycles in his various writings, but particularly in *Twelve-Tone Tonality* (1996). They consist of interlocked, inversionally related interval cycles.

[8] Klumpenhouwer networks and their isographies first appeared in Lewin (1990), though the concept originates with Klumpenhouwer (1991). Perle (1993) suggests that we may consider all Klumpenhouwer networks as segments of Perle cycles (or, rather, Perle-Lansky cycles), a fact that Lewin (2002) demonstrates. Each of the four articles in *Music Theory Spectrum* 24.2 (Lambert 2002, Lewin 2002, Stoeker 2002, and Headlam 2002) focuses on the study of Klumpenhouwer networks and their relationship to Perle- and Perle-Lansky cycles. See also Morris (1998, 191–93) for a related discussion.

[9] An isomorphism is a one-to-one mapping F of a group G onto another H such that $F(a)F(b) = F(ab)$ for any elements a and b in G. F and H, then, have the same algebraic structure. In an anti-isomorphism, we also have a one-to-one mapping F′ of a group G onto another H′. Now, however, $F'(a)F(b) = F'(ba)$. Even though the ordering of a and b reverses on opposite sides of this equation, the algebraic structure of G and H′ is indeed the same. Anti-isomorphisms are discussed in Lewin (1987, 14) and Robinson (1982, 216).

[10] The notation "T/I" is often used for the order 24 dihedral group of T_n and I_n operators in the music theoretical literature. It does not signify a quotient group, as the notation might suggest.

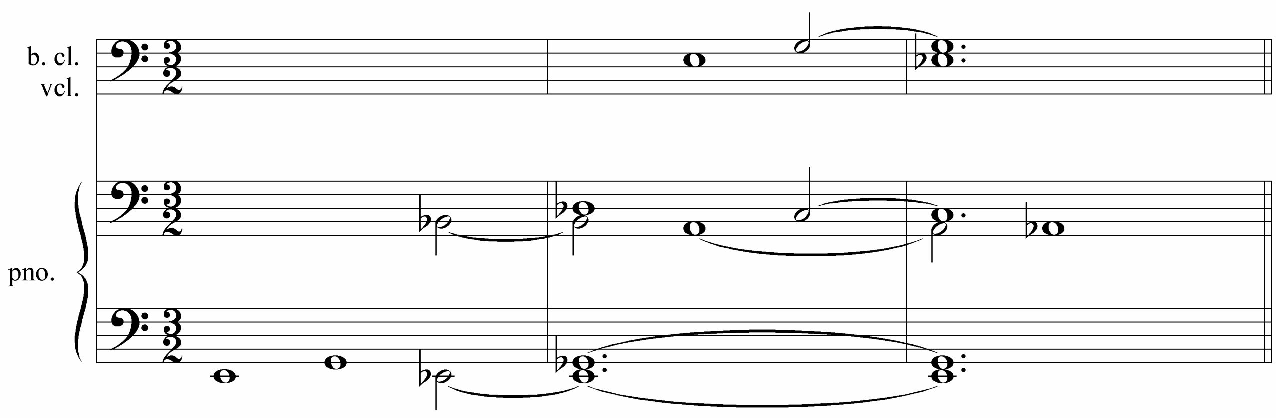
[11] For instance, see Lewin's (2002, 201) Example 2.8.

[12] Lambert (2002) investigates the occurrences of various set-classes in Klumpenhouwer-network classes (K-classes, following O'Donnell 1998).

[13] The anti-isomorphism of Perle cycles to T/I Klein-bottle *Tonnetze* thus recalls Lewin's (1987, 46–48) anti-isomorphism of "interval-preserving operations" to "transposition."

[14] For example, in m. 8, the bass clarinet plays three occurrences of 3-3[014]: {E,G,E♭}, {G,B♭,G♭}, and {E♭,G♭,D}. The pitch-classes of each individual trichord may be interpreted using (T_3,T_8,T_1). Furthermore, these same operators also relate the successive trichords recursively as pitch-class sets.

Example 1a. Schoenberg, no. 8, "Nacht," from *Pierrot Luinaire*, mm. 1-3 (sounds 8va bassa)



Example 1b. Imbricated trichords in "Nacht," mm. 1-3

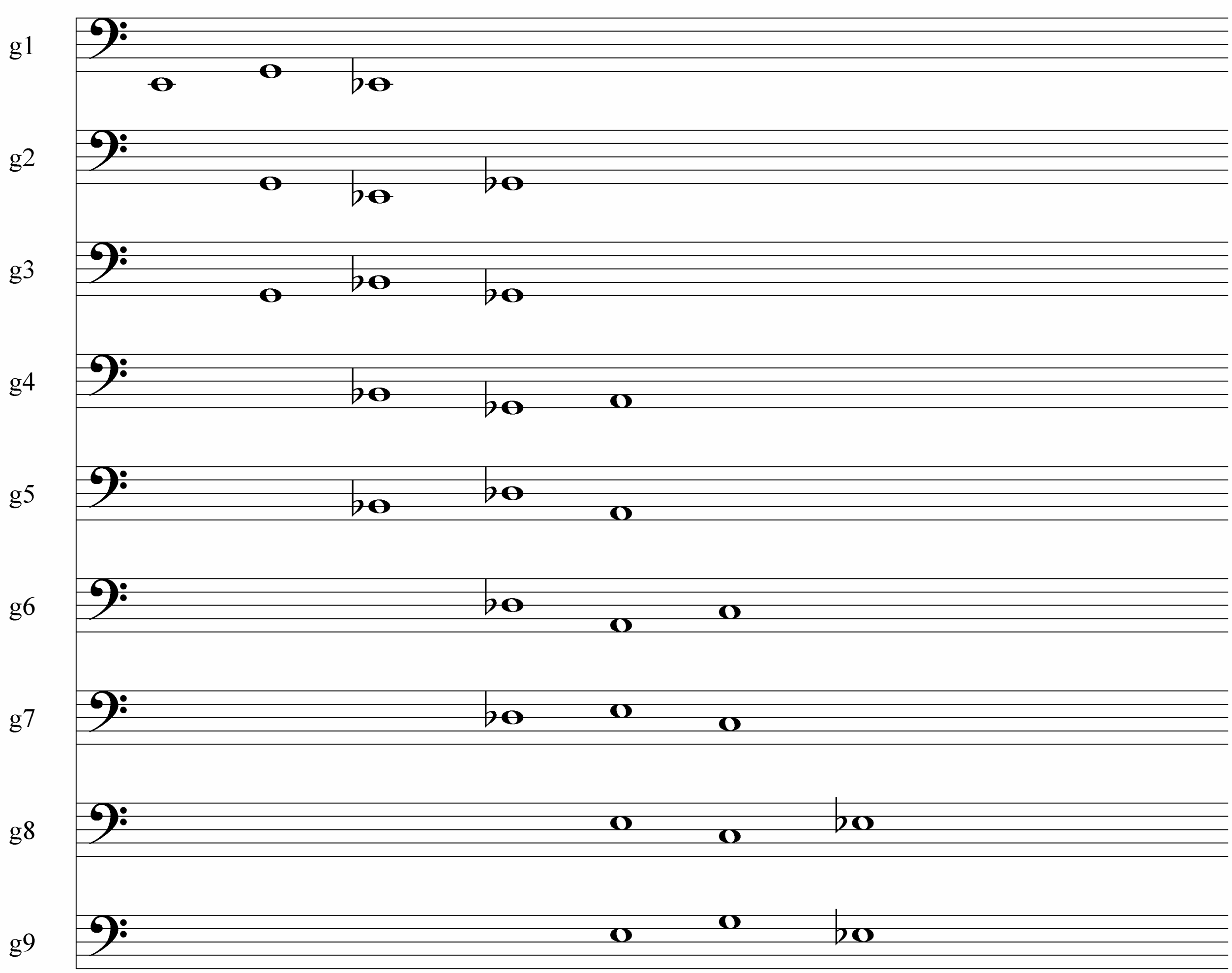


Figure 2. Network of imbricated trichords in the opening canon of "Nacht"

```
      3<---T_9_---6<---T_9_---9<---T_9_---0<---T_9_---3
     / \   g2    / \    g4   / \    g6   / \   g8    / \
    /   \       /   \       /   \       /   \       /   \
  I_7_  I_10_ I_1_  I_4_  I_7_  I_10_ I_1_  I_4_  I_7_  I_10_
  /       \   /       \   /       \   /       \   /       \
 /   g1    \ /   g3    \ /   g5    \ /   g7    \ /  g9=g1  \
4----T_3_-->7----T_3_-->10---T_3_-->1----T_3_-->4----T_3_-->7
```

[1.5] Figure 3 places the above trichordal subnetworks into a supernetwork, in which nodes represent underlying graphs, and edges represent hyper-operators.[15] It is also a type of Klein-bottle network. The recursion between the two networks is evident when we compare the operators of Figure 2's edges with the hyper-operators of the edges of Figure 3. Regardless of node content—hence, taken only as graphs—we find a direct correspondence among these edges. The same operators which relate pitch-classes in Figure 2 also relate, via conjugation, the operators that interpret its trichords.[16] Thus, we find a remarkable degree of consistency among the various levels of the example.

Figure 3. Supernetwork of trichordal subnetworks in Figure 2

```
      g5<-[T_9_]--g7<-[T_9_]--g1<-[T_9_]--g3<-[T_9_]--g5
     / \         / \         / \         / \         / \
    /   \       /   \       /   \       /   \       /   \
[I_7_][I_10_][I_1_][I_4_][I_7_][I_10_][I_1_][I_4_][I_7_][I_10_]
  /       \   /       \   /       \   /       \   /       \
 /         \ /         \ /         \ /         \ /         \
g4--[T_3_]->g6--[T_3_]->g8--[T_3_]->g2--[T_3_]->g4--[T_3_]->g6
```

[1.6] In the following sections, we generalize the theory of Klein-bottle *Tonnetze*. First, we arrive at an algebraic and graph-theoretical abstraction, and consider the relation of Klein-bottle *Tonnetze* to the more familiar toroidal models. Then, we examine various pitch-class networks that possess these properties. Specifically, we focus on those Klein-bottle *Tonnetze* which incorporate the cyclic T group, the dihedral T/I group, and a generalized quaternion subgroup of the T/M group.[17] We examine next the various isographies that relate entire Klein-bottle *Tonnetze* to one another. Finally, we make some connections between neo-Riemannian and Klein-bottle *Tonnetz* theories, and suggest some further areas for musical analysis.

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[15] Figure 3 uses hyper-T_n- and hyper-I_n-operators that derive from the inner automorphism group of the T/I group. $[T_n]$ describes a conjugation of the T/I group by T_n; $[I_n]$ is a conjugation by I_n. We follow Klumpenhouwer's (1998) convention of using square brackets for these hyper-operators, and angle brackets for those deriving from the full automorphism group.

[16] These same operators also relate the trichords as pitch-class sets, a special property of hyper-operators that derive from the inner automorphism group. See Klumpenhouwer (1998).

[17] A quaternion group Q_{2^n}, where $n \geq 3$, has a presentation of $\langle x,y \mid x^{2^{(n-1)}} = e, y^2 = x^{2^{(n-2)}}, yxy^{-1} = x^{-1}\rangle$. The order of the group is, accordingly, a power n of 2 (Robinson 1982, 136). However, the T/M subgroup we will incorporate is not a true quaternion group. Rather, it is one of a class of so-called generalized quaternion groups Q_{4n}, with a presentation $\langle x,y \mid x^{2n} = e, y^2 = x^n$, and $yxy^{-1} = x^{-1}\rangle$ (from personal correspondence with Edward Gollin).

[2] KLEIN-BOTTLE GRAPHS

[2.1] Theoretically, Riemann's table in Figure 1 extends infinitely on a plane in two dimensions. Using its underlying graph, we may define these dimensions as follows: translation by x moves each node to the right by one, and translation by y moves each node upward by one. These translations correspond to musical transpositions by a just perfect fifth and a just major third, respectively. In other words, we map the figure's interval content onto the infinite cyclic groups X and Y. By assigning finite orders to X and Y—musically, by accepting enharmonic and octave equivalence—we identify the figure's two sets of parallel edges. Now each of the axes on the plane is circularized, and the figure forms a grid on the surface of a torus. The transformation group for the graph is given by the product set XY, and, because X and Y are both groups of translations, XY is also cyclic group, hence commutative.[18] Accounting for node content, this group is generated by T_7 and T_4, which yields the familiar cyclic T group.[19]

[2.2] Given a particular relation, we observe another geometry which arises from the product of two cycles: the Klein bottle. Initially, a Klein bottle is constructed like a torus. If we start with a rectangle, and bend it to identify two parallel edges, we obtain an open cylinder. If we then bend the cylinder around to join its two ends, we get a torus. To make a Klein bottle, we need to return to the cylinder. Again, we join its two ends, but not by bending the cylinder around; rather, we thrust one end through the side of the cylinder, and connect it to the other end internally. The surface must not really intersect itself through where the bottle's neck is thrust. Rather, a fourth dimension is needed to go around the surface instead of through it. The result is an unbound surface with no inside or outside.

[2.3] Figure 4 shows a pair of rectangles, and the identifications of their sides that yield respectively a torus and a Klein bottle.[20]

Figure 4. Fundamental regions of a two-dimensional torus and Klein bottle

```
a) torus                          b) Klein bottle
    y----->-----xy = yx               w----->-----xw = wx^-1^
    |           |                     |           |
    ^           ^                     ^           ^
    |           |                     |           |
    e----->-----x                     e-----<-----x
```

In Figure 4a, we identify the edges indicated by the ordered pairs (e,y) and (x,yx) by taking e to x, and y to xy, forming an upright cylinder. Then we identify edges (e,x) and (y,xy) by taking e to y and x to xy, obtaining a torus. In Figure 4b, we identify (e,w) and (x,xw) by taking x to e and w to xw. Again, we have an upright cylinder. Now we identify (e,x) with (xw,w) by taking e to xw and x to w, thus forming a Klein bottle.

---

[18] We may give the group of the torus as $G = \langle x,y \mid x^m = y^n = e, xy = yx\rangle$. It is a quotient of the plane symmetry group generated by two translations. We use the letter "e" for the identity element.
[19] In fact, the full T group may be generated by T_7 alone; T_4's presence as a generator here is redundant. A more satisfying two-dimensional network in pitch-class space may be generated by T_3 and T_4, which intersect trivially, and also yield the full group.
[20] In the present study, we use left-functional orthography, following the standard music theoretic notation for the T/I group. In other words, the composition "yx" means "do x first, then to y."

[2.4] To arrive at the relations necessary for a Klein bottle group, we start with two distinct cyclic groups W and Z, generated by w and z, respectively. On the underlying graphs of our *Tonnetze*, we define the operations w and z not as linear translations, like x and y, but rather as parallel glide reflections.[21] A glide reflection is the product of two operations: first it reflects in an axis, then translates parallel to that axis.[22] Figure 5 shows a fragment of a W-cycle of glide reflections.

Figure 5. A portion of the W-cycle of glide reflections

```
  w^3^
   \|
    \
    |\
    | w^2^
    |/
    /
   /|
  w | +
   \|
    \
    |\
  * | e
    |/
    /
   /|
 (w^-1^)
```

From e, w reflects first across an axis parallel to Y, taking e to *. Then w translates upward, taking * to w. This combination completes a move by w. A subsequent move by w also reflects across the same axis, now taking w to +, and translates upward, taking + to w^2, and so forth. Of course, it is important to keep in mind that, like translations, glide reflections act on the entire plane, and not just on a single point.

[2.5] Figure 6 shows a portion of another cycle of glide reflections, this one generated by z, together with our previous W-cycle. A move by z reflects across a different axis, parallel to Y, and then translates upward. Hence, both operations are sense-reversing with regard to a vector pointing in the direction of the (horizontal) X-axis.

We note the following important relation,[23]

DEFINITION 2.5.1 $w^2 = z^2$,

which, for a finite w and z, incorporates into a presentation of a Klein-bottle group, G.

DEFINITION 2.5.2 $G = \{w,z \mid w^m = z^n = e, w^2 = z^2\}$.

---

[21] Glide reflections are discussed further in Coxeter and Moser (1965, 56) and Escher (1967, 12–13).
[22] These operations of reflection and translation (parallel to the axis of reflection) commute, so we achieve the same result by translating first, then reflecting.
[23] Coxeter and Moser (1965, 42–43) give an infinite plane symmetry group "p g," which they present using two parallel glide reflections, $w^2 = z^2$, as in our 2.5.1 (however, they use variable names P and Q). By assigning finite orders to w and z, we obtain a quotient of the infinite group. This quotient is a Klein-bottle group.

Figure 6. A portion of a Z-cycle beginning at e

```
 w^3^  | z^3^
  \    |/
   \   /
    \ /|
     z^2^=w^2^
    / \|
   /   \
  /    |\
 w   + | z
  \    |/
   \   /
    \ /|
     e | *
    / \|
   /   \
  /    |\
(w^-1^)|(z^-1^)
```

[2.6] The product of any two glide reflections is a translation. Accordingly, we may now define the translations x and y above in terms of w and z.

DEFINITION 2.6.1 $x = z^{-1}w$.

DEFINITION 2.6.2 $y = w^2 = z^2$.

Figure 7 is an illustration of e, w, x, y, and z, all originating from e.

Figure 7. Elements e, w, x, y, and z of the Klein-bottle group

```
     y = w^2^ = z^2^
    /|\
   w | z
    \|/
     e---x = z^-1^w
```

Using 2.6.1-2, we note further that

THEOREM 2.6.3 $w = zx$

and

THEOREM 2.6.4 $z = wx^{-1}$.[24]

We may also give an alternative definition of z:

COROLLARY 2.6.5 $z = xw$.

[24] The proofs of all theorems in the text appear in the appendix ([9]).

[2.7] Now we construct an appropriate fundamental region for our Klein-bottle group. Whereas the relation in 2.5.1 suggests Figure 8, its graph does not intuitively resemble a Klein bottle.

Figure 8. Fundamental region based on 2.5.1

```
     y
    / \
   ^   ^
  /     \
 w--->---z
  \     /
   ^   ^
    \ /
     e
```

Therefore, we will construct a variant that is easier to visualize. In doing so, we will move the top half of the figure by some member of the group.[25] Multiplying the nodes of the upper triangle, (w,z,y), on the left by z^{-1} gives (x,e,z).

TABLE 2.7.1 Mapping of (w,z,y) onto z^{-1}(w,z,y) = (x,e,z)

```
Member of        Member of
(w,z,y)    z^-1^ (x,e,z)
-------------------------
  w        --->  x  (by 2.6.1)
  z        --->  e  (by cancellation)
  y        --->  z  (by 2.6.2)
```

Next, we reconstruct the diagram using the original bottom half of Figure 8 and this variant of its top half (Figure 9). This diagram is also a fundamental region.

Figure 9. Reconstructed fundamental region

```
w--->---z
 \     / \
  ^   ^   ^
   \ /     \
    e---<---x
```

Now we can more readily visualize the Klein bottle, using the following identifications of sides: (e,w) → (x,z) (by x), and (w,z) → (x,e) (by z^{-1}).

[2.8] A Klein-bottle group may be generated equivalently by using either two glide reflections, as above, or by one glide reflection and one translation. Since the latter conforms more to our notions

[25] The practice of cutting off part of a fundamental region, and adding a congruent piece elsewhere by some member of the group, is standard in the mathematical literature. See Coxeter and Moser (1965, 44–45).

of the dihedral T/I group, we observe the following relations:[26]

DEFINITION 2.8.1 $w^2 = y; xy = yx; w(x)w^{-1} = x^{-1}$

that incorporate into a presentation of a group G in terms of its generators w and x.

DEFINITION 2.8.2 $G = \langle w,x \mid w^2 = y; xy = yx; w(x)w^{-1} = x^{-1}; w^m = x^n = e \rangle$.

Given no further relations, we note that G is non-commutative.

[2.9] In the generalized group, only the members of subgroup Y commute always with every member of the group. Therefore,

THEOREM 2.9.1 *Y is in the center of G.*

Accordingly, it is easy to show that Y is also a normal subgroup. Furthermore,

THEOREM 2.9.2 *X is a normal subgroup of G;*

and,

THEOREM 2.9.3 *if X has an even order k, then* $x^{k/2}$ *is in the center of G.*

Now we may define the center of G, C_G.

DEFINITION 2.9.4 *If* $2 \mid |X|$, *then* $C_G = \langle y, x^{|X|/2} \rangle$; *otherwise,* $C_G = \langle y \rangle$.

[2.10] Specifically, because X is normal in G, we may offer an alternate definition of G to 2.8.2,

DEFINITION 2.10.1 $G = XW$,

using the product formula for the set XW. The order of G is thus determined:

DEFINITION 2.10.2 $|G| = |W||X| \,/\, |W \cap X|$.

Furthermore,

THEOREM 2.10.3 $2 \mid |G|$.[27]

Figure 10 shows a larger segment of an abstract Klein-bottle *Tonnetz*. Given no further relations, we find 2|G| unique triangles formed by mutually adjacent nodes in such a graph.

---

[26] The relations in 2.8.1 present the same infinite group as in 2.5.1.

[27] As a consequence of 2.10.3, no Klein-bottle *Tonnetze* exist in mod 7 diatonic space, or in any other spaces of odd orders.

Figure 10. A portion of an abstract Klein-bottle *Tonnetz*

```
x^-2^w^2^___x^-1^w^2^_w^2^_____xw^2^
      \      / \      / \      / \
       \    /   \    /   \    /   \
        \  /     \  /     \  /     \
        x^-1^w___w_________xw_______x^2^w
           \      / \      / \      / \
            \    /   \    /   \    /   \
             \  /     \  /     \  /     \
             x^-1^____e_________x_________x^2^
                \      / \      / \      / \
                 \    /   \    /   \    /   \
                  \  /     \  /     \  /     \
                  w^-1^___xw^-1^___x^2^w^-1^___x^3^w^-1^
```

[2.11] In certain circumstances, G may be commutative, and in any such Klein-bottle *Tonnetz*, x generates an involution.

> THEOREM 2.11.1 *If ab = ba for any a,b in G, then* $|X| = 2$.

For example, all Klein-bottle *Tonnetze* that use the commutative T group will have $X = \langle T_6 \rangle$, the only involution in that group.

[2.12] Finally, we note the condition under which the groups of two Klein-bottle *Tonnetze*, G and G′, are isomorphic. This condition will be of particular consequence in later sections. We define the isomorphism in terms of a bijective mapping, F, of G onto G′.

> DEFINITION 2.12.1 *Let G and G′ be two groups, and let F be a bijective set mapping F: F(G) = G′. F is an isomorphism of G to G′ if F(h)F(g) = F(hg), for any g,h in G.*

Certain groups are isomorphic to themselves (by a map other than the identity), and such internal relationships define (nontrivial) automorphisms.[28]

> DEFINITION 2.12.2 *Let G be a Klein-bottle group, and let F be a bijective set mapping F: F(G) = G. F is a group automorphism if, for any g,h in G, F(h)F(g) = F(hg).*

[2.13] The opening of Witold Lutosławski's *Funeral Music* (Example 2) provides an illustration of how we can map musical materials to the nodes of a Klein-bottle graph. The piece begins with a twelve-tone row, P_0, presented in the Cello I solo. It consists of a fragment of a cycle of alternating tritones and descending minor seconds. This row is followed immediately in the same voice by a statement of I_6, then another statement of P_0, and so forth. The Cello II solo enters in strict imitation at the half note with a P_6 form of the row. It continues correspondingly with I_0, and again P_6. Because of the row's intervallic construction, every other harmonic interval in this canon is either a unison or an octave.

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[28] These automorphisms are not necessarily *inner* automorphisms, given by the mapping F: $F(G) = g(G)g^{-1}$, for some g in G. However, in certain circumstances, they may be inner, as we will see below. For a discussion of inner and outer automorphisms in the context of the T/I group, see Klumpenhouwer (1998).

Example 2. Lutoslawski, *Funeral Music*, I ("Prologue"), mm. 1-8

[2.14] The full cycle of alternating T_{11}'s and T_6's, in that order, is twenty-four units long. We can map its operators onto the elements of an order 24 cyclic group W, as shown in Figure 11.

Figure 11. Mapping of cycle of alternating descending minor seconds and tritones onto a cyclic group W

```
w^1^  w^2^  w^3^  w^4^  w^5^  w^6^  w^7^  w^8^  w^9^  w^10^ w^11^ w^12^
T_11_ T_5_  T_4_  T_10_ T_9_  T_3_  T_2_  T_8_  T_7_  T_1_  T_0_  T_6_

w^13^ w^14^ w^15^ w^16^ w^17^ w^18^ w^19^ w^20^ w^21^ w^22^ w^23^ w^24^
T_5_  T_11_ T_10_ T_4_  T_3_  T_9_  T_8_  T_2_  T_1_  T_7_  T_6_  T_0_
```

Next, we can produce another order 24 cycle by alternating T_6's and T_{11}'s, in that order. We map its elements onto a cyclic group Z. (See Figure 12.)

Figure 12. Mapping of cycle of alternating descending minor seconds and tritones onto a cyclic group Z

```
z^1^  z^2^  z^3^  z^4^  z^5^  z^6^  z^7^  z^8^  z^9^  z^10^ z^11^ z^12^
T_6_  T_5_  T_11_ T_10_ T_4_  T_3_  T_9_  T_8_  T_2_  T_1_  T_7_  T_6_

z^13^ z^14^ z^15^ z^16^ z^17^ z^18^ z^19^ z^20^ z^21^ z^22^ z^23^ z^24^
T_0_  T_11_ T_5_  T_4_  T_10_ T_9_  T_3_  T_2_  T_8_  T_7_  T_1_  T_0_
```

Under this mapping, $w^2 = z^2$, as $T_6T_{11} = T_{11}T_6 = T_5$. Furthermore, $w^{24} = z^{24} = e$. Thus, all the relations for a Klein-bottle group from 2.5.2 are satisfied.

[2.15] Successive application of the members of W and Z to the pitch-class 6 yields the following two cycles of pitch-classes:

W(6) = (6,5,11,10,4,3,9,8,2,1,7,6,0,11,5,4,10,9,3,2,8,7,1,0) and
Z(6) = (6,0,11,5,4,10,9,3,2,8,7,1,0,6,5,11,10,4,3,9,8,2,1,7).

Alternating members of these respective cycles belong to the same pitch-class. The Cello I solo presents a twelve-member fragment of W(6), beginning on the pitch-class 5 in the cycle's second position: (5,11,10,4,3,9,8,2,1,7,6,0). This line continues with a twelve-member fragment from the retrograde of Z(6): (11,5,6,0,1,7,8,2,3,9,10,4). The Cello II solo imitates the Cello I solo with a (non-retrograded) fragment of Z(6), beginning on the pitch-class 11 in the cycle's third position: (11,5,4,10,9,3,2,8,7,1,0,6). It continues with a fragment from the retrograde of W(6): (5,11,0,6,7,1,2,8,9,3,4,10).

[2.16] These twelve-tone rows can be modeled entirely using a closed segment from the Klein-bottle *Tonnetz* generated from this W(6) and Z(6). Figure 13 presents this fragment. The Cello I solo begins with the pitch-class 5 in the figure's lower left-hand corner, and proceeds upward with pitch-classes 11 and 10 in the the figure's middle and leftmost columns, respectively. The line continues with pitch-classes 4, 3, 9, 8, 2, 1, 7, 6, and 0, maintaining the alternation between these two columns. Next, it presents pitch-classes 11, 5, and 6, rounding the uppermost portion of the figure, and it concludes with pitch-classes 0, 1, 7, 8, 2, 3, 9, 10, and 4, descending and alternating between the rightmost two columns. At this point, it reaches pitch-class 5 again, and then repeats the entire cycle, and so forth. In short, it describes a jagged clockwise path around the figure. The Cello II solo begins with the pitch-class 11 at the bottom of the middle column, and it proceeds around the diagram in a jagged counterclockwise path.

Figure 13. Klein-bottle *Tonnetz* fragment

```
    5
   / \
 11   6
   \ /
    0
   / \
 6    1
   \ /
    7
   / \
 1    8
   \ /
    2
   / \
 8    3
   \ /
    9
   / \
 3    10
   \ /
    4
   / \
 10   5
   \ /
    11
   /
(5)
```

[3] KLEIN-BOTTLE NETWORKS

[3.1] We will now formalize the previous example. We will take G as a group of operations, and will examine the network obtained by applying all the members of G to a common object, p. For example, if p is a pitch-class integer, we obtain a Klein-bottle *Tonnetz*. We start with the concept of a P-set.

DEFINITION 3.1.1 $P = G(p)$.

In other words, P is the orbit of p under G. We determine the order of P by extension of 2.10.2.

THEOREM 3.1.2 $|P| = |W(p)||X(p)| / |W(p) \cap X(p)|$.[29]

In all cases, $|X(p)| = |X|$. Being a group of translations, X is fixed-point-free. However, in certain circumstances, $|W(p)| < |W|$, since W, as a group of glide reflections, may or may not be fixed-point-free.[30] Furthermore, sometimes $|W(p) \cap X(p)| > |W \cap X|$. This situation may occur in contexts in which certain images of p under both W and X are not unique.[31] Moreover, because of the closure of G under its group operation, applying G to any member of P produces the same P-set.

REMARK 3.1.3 $G(p_i) = G(p_j)$ *for any* p_i, p_j *in P.*

[3.2] In all events, our *Tonnetz* is now a true GIS in the sense of Lewin (1987, 26-30). We may represent it by the ordered triple (S,IVLS,int), in which the space S of the GIS consists of P; the group of intervals IVLS acting on P is G; and int is a function which assigns a value g from G to any pair (p_i,p_j) in $P \times P$. In other words, $\mathrm{int}(p_i,p_j) = g$, for some g in G. Our *Tonnetz* is, moreover, transitive, and also satisfies Lewin's Condition (B) for uniqueness: for every p_i in P and g in G, there is a unique p_j in P which lies the interval g from p_i.

[3.3] We will now examine a segment from an abstract operational *Tonnetz* (Figure 14).

Figure 14. A segment from an abstract Klein-bottle *Tonnetz*

```
        e(p_4_)
          / \
       w ^   ^ z
        /     \
       /   x   \
e(p_2_)-->--e(p_3_)
       \       /
      w ^     ^ z
         \   /
          \ /
        e(p_1_)
```

[29] The lengthy proof of 3.1.2 is omitted in the appendix for reasons of space. The interested reader may wish to begin with the notion of X's being a normal subgroup in G (using 2.9.2). Equivalent to 3.1.2, we may also use the orbit-length formula to determine the size of P: $|P| = |G| / |G_P|$, where G_P is the stabilizer in G of P. See Robinson (1982, 31).

[30] A simple example may be found using $w = I_0$. Then, $w(6) = 6$, and $w^2(6) = e(6) = 6$, the same point. Consequently, in this case, $|W(p)| = 1$ is less than $|W| = 2$.

[31] For example, using generators $w = I_8$ and $x = T_2$, $(W \cap X) = \{T_0\}$ consists of a single member. Nevertheless, applied to a pitch-class $p = 0$, $W(0) = (0,8)$, $X(0) = (0,2,4,6,8,10)$, and $(W(0) \cap X(0)) = (0,8)$. We now find a two-member intersection set.

Following Lewin, Figure 14 is a network. Its space, which consists of the set of points S = $\{p_1,p_2,p_3,p_4\}$, is a subset of P. The intervals between these points, $\{e,w,w^{-1},x,x^{-1},y,y^{-1},z,z^{-1}\}$, form a subset of IVLS = G. Table 3.3.1, then, gives the specific mapping of S × S into G.

TABLE 3.3.1 Mapping int(p_i,p_j) of S into G

(p_i_,p_j_)	Interval	(p_i_,p_j_)	Interval
(p_1_,p_1_)	e	(p_3_,p_1_)	z^-1^
(p_1_,p_2_)	w	(p_3_,p_2_)	x^-1^
(p_1_,p_3_)	z	(p_3_,p_3_)	e
(p_1_,p_4_)	y	(p_3_,p_4_)	z
(p_2_,p_1_)	w^-1^	(p_4_,p_1_)	y^-1^
(p_2_,p_2_)	e	(p_4_,p_2_)	w^-1^
(p_2_,p_3_)	z	(p_4_,p_3_)	z^-1^
(p_2_,p_4_)	w	(p_4_,p_4_)	e

[3.4] Next, for any p_i in P, we observe two classes of triangles, t+(p_i) and t-(p_i), formed by mutually adjacent nodes on the graph of G(p);[32] t+ triangles point upward and t- triangles point downward. We define the set of all such triangles,

DEFINITION 3.4.1 *TRI = {t+(p_i),t-(p_i) | p_i is a member of P}*

and assign only one triangle of each type to any node p_i in the graph.

DEFINITION 3.4.2|*TRI*| = 2|*P*|.

The node p_i of origin serves as the "*Einheit*" of the triangle, which we abbreviate "h." We describe the *Einheit* as being either even or odd, based on the parity of the least power of w (from e) that determines it.[33]

DEFINITION 3.4.3 *h is even if it is determined by an even least power of w.*

DEFINITION 3.4.4 *h is odd if it is determined by an odd least power of w.*

Therefore, we may define an equivalence relation which partitions all such triangles into four equivalence classes.

---

[32] These triangles are generally not Cohn functions in the sense of Lewin (1996, 182–84). Whereas rotated retrogrades of the interval series obtain by exchanging adjacent arguments in (at least) two different locations, the resultant series no longer describe closed (N.B.) shapes on the *Tonnetz*. For instance, using the T/I group, the interval series (T_7,I_{11},I_4) originating on pc 0 describes a C major triad. Taken as a composition, this series yields the identity element: $I_4I_{11}T_7 = T_0$. Now, by exchanging the first two arguments, we derive (I_{11},T_7,I_4). Its corresponding composition does not equal T_0; rather, $I_4T_7I_{11} = T_{10}$. (The compositions of the remaining two flips also yield non-identity elements.) In other words, this failed Cohn flip does not return us to our point of origin, pc 0. The triangles are Cohn functions only in instances in which x generates an involution, as in all commutative G's, including those that use T operators exclusively. These may be Cohn-flipped in (at least) two different ways.

[33] In 3.4.3-4, we give the condition "least power of w" because, in certain G's, any particular p_i may be determined by both odd and even powers of w. For example, let |W| = 5. Now, if $p_i = w^3(p)$, then $p_i = w^8(p)$ is also true. Therefore, as an *Einheit*, we say that this p_i is odd, as it is determined by an odd least power of w, w^3. Such situations demonstrate the one-sided topology of the Klein-bottle.

DEFINITION 3.4.5 *Let* TRI *be the relation: “points in the same direction as, and has the same Einheit parity as.”*

We label the four classes of triangles as follows:

DEFINITION 3.4.6 *The four TRI-classes of triangles: t+/e, t+/o, t-/e, t-/o.*

Figures 15a-b present these triangles for even and odd *Einheiten*, respectively; all even *Einheiten* appear on the left-hand side of the triangles, and all odd ones on the right.

Figure 15. The four *TRI*-classes of triangles in Klein-bottle *Tonnetze*

```
a) Triangles t+/e and t-/e        b) Triangles t+/o and t-/o

            /\                                /\
           v  ^                              ^  v
          /    \                            /    \
         / t+/e \                          / t+/o \
        h--->----                          ---<----h
         \ t-/e /                          \ t-/o /
          \    /                            \    /
           ^  v                              v  ^
            \/                                \/
```

[3.5] In general, each of the four types of triangles is determined by a unique mod 3 function.[34]

DEFINITION 3.5.1 *t+/e =* $(x,x^{2(j+1)}w,x^{2(j+1)-1}w^{-1})$ *: j = the power of x that determines h.*

DEFINITION 3.5.2 *t+/o =* $(x^{-1},x^{2(j-1)}w,x^{2(j-1)+1}w^{-1})$ *: j = the power of x that determines h.*

DEFINITION 3.5.3 *t-/e =* $(x,x^{2(j+1)}w^{-1},x^{2(j+1)-1}w)$ *: j = the power of x that determines h.*

DEFINITION 3.5.4 *t-/o =* $(x^{-1},x^{2(j-1)}w^{-1},x^{2(j-1)+1}w)$ *: j = the power of x that determines h.*

Furthermore, the interval series of any triangle determines its total GIS-interval content.

DEFINITION 3.5.5 *Let (a,b,c) be the interval series of a triangle. Then,* $(e,a,a^{-1},b,b^{-1},c,c^{-1})$ *is the total GIS-interval content for that triangle.*

---

[34] Following Gollin (1998, 199–200), we consider, in a neo-Hauptmannian sense, the three nodes of any one of these triangles as the *Einheit*, *Zweiheit*, and *Verbindung*. For example, let p_i be the *Einheit* of a t+/e triangle. Now, 3.5.1 gives $x(p_i)$ as the *Zweiheit*, and $x^{2(j+1)}w(x(p_i))$ as the *Verbindung*. Then, $x^{2(j+1)-1}w^{-1}(x^{2(j+1)}w(x(p_i))) = e(p_i)$ is again the *Einheit*. This procedure may also be used for the other three classes of triangles, using 3.5.2-4.

It is then easily demonstrated that any pair of t+ and t- triangles that share the same *Einheit* possess the same GIS-interval content.

> THEOREM 3.5.6 *For any* Einheit *h, t+(h) and t-(h) have the same GIS-interval content.*

[3.6] In general, the various interval series of triangles in the same *TRI*-class vary with the powers of x that determine their *Einheiten*, except among triangles that relate by operations in the center of G.

> THEOREM 3.6.1 *Triangles of the same* TRI-*class whose nodes are (left) transforms of each other by operations in* C_G *(2.9.4) have the same interval series.*

Therefore, in commutative G's, in which the center of G is G itself, all t+ triangles have the same interval series, as do all t- triangles; hence, by 3.5.6, all triangles in TRI have the same GIS-interval content. In non-commutative G's, triangles of the same class that are not transforms of each other by operations in the center of G still relate significantly. As we will see in [5], when using the T/I group, they form weakly isographic Klumpenhouwer networks.

[3.7] Another condition also gives rise to equivalent interval series among triangles of different *TRI*-classes. When w generates an involution, as in cases wherein w is an inversion operator, t+ and t- triangles sharing the same *Einheit* display the same interval series. Clearly, then, in cases in which both x and w generate involutions, such as when G is a four group, all triangles in TRI have the same interval series.

[3.8] Before leaving the topic of interval series, we explore one more situation: instead of preserving a congruent shape—as with triangles formed by mutually adjacent nodes above—and varying p_i, we will now study the results of preserving an interval series. This situation recalls Lewin's (1987, 46-48) distinction between "transposition" and "interval-preserving operations," and leads to a corresponding anti-isomorphism. In general, two segments with identical interval series are not congruent, as Figures 16a-b demonstrate for two segments with the following interval series (x,x^2w,xw^{-1}). The segments are congruent only when one relates to the other by some operation in the center of G.

Figure 16. Two segments that use interval series (x,x^2w,xw^{-1})

```
a)           xw(p)               b)   w(p)
            / \                          \    \
           /   \                          \     \
          /     \                          \      \
         /       \                          \       \
        e(p)------x(p)                     x(p)------x^2^(p)
```

[3.9] We may, however, reconfigure the nodes to preserve congruency. Figure 12 shows a segment of a *Tonnetz* in which every t+ triangle has interval series $(x,w,x^{-1}w^{-1})$, and every t- triangle has $(x,w^{-1},x^{-1}w)$, regardless of their *Einheiten* parities.[35] We will call this network H(p), which derives

[35] Then, by 3.5.6, all triangles in Figure 12 have the same GIS-interval content.

from a particular group automorphism F of G. (Compare Figures 17 and 10.) Using the generators w and x of G and their product, Table 3.9.1 gives the mapping F of G(p) onto H(p), which indicates the anti-homomorphism condition F(ab) = F(b)F(a) for any a,b in G.[36]

Figure 17. A portion of H(p)

```
x^-2^w^2^(p)__x^-1^w^2^(p)__w^2^(p)
 \           / \           / \
  \         /   \         /   \
   \       /     \       /     \
    \     /       \     /       \
     \   /         \   /         \
      \ /           \ /           \
       xw(p)_________w(p)__________x^-1^w(p)
        \           / \           / \
         \         /   \         /   \
          \       /     \       /     \
           \     /       \     /       \
            \   /         \   /         \
             \ /           \ /           \
              x^-1^(p)______e(p)__________x(p)
               \           / \           / \
                \         /   \         /   \
                 \       /     \       /     \
                  \     /       \     /       \
                   \   /         \   /         \
                    \ /           \ /           \
                     w^-1^(p)______x^-1^w^-1^(p)__x^-2^w^-1^(p)
```

TABLE 3.9.1 Anti-isomorphism of G(p) with H(p)

Member of G(p)	F	Member of H(p)
w(p)	-->	w(p)
x(p)	-->	x(p)
xw(p)	-->	x^-1^w(p) = wx(p) (by extension of 2.6.4-5)

[3.10] We will now discuss certain potential relationships among abstract Klein-bottle *Tonnetze*. We recall the definition of isomorphic groups G and G′ in 2.12.1. We now apply that definition to our operational networks.

> DEFINITION 3.10.1 *Let G´(p´) and G(p) be Klein-bottle* Tonnetze*, and let F be a bijective mapping F: F(G´(p´)) = G(p); p´ may or may not be a member of G(p). F is an isography of G´(p´) onto G(p) if G´ is isomorphic to G.*

[36] When using $w = I_m$ and $x = T_n$, this anti-isomorphism is precisely that between Klein-bottle *Tonnetze* and Perle cycles, to which we alluded in [1.2].

It is important to emphasize that such mappings preserve not only group structure, but also graph structure: they take vertices to vertices, and edges to edges.

[3.11] Following 2.12.2, certain G's are non-trivially isomorphic to themselves. In terms of these group automorphisms, we may define group-automorphic isographies, or GA-isographies.

> DEFINITION 3.11.1 *Let $G'(p')$ and $G(p)$ be Klein-bottle* Tonnetze*, and let F be a bijective mapping F: $F(G'(p')) = G(p)$; p' may or may not be a member of $G(p)$. F is a GA-isography of $G'(p')$ onto $G(p)$ if $G' = G$.*

Finally, we note a special case of GA-isography, in which p′ is a member of G(p). This F is a graph automorphism, or autography.

> DEFINITION 3.11.2 *Let $G'(p')$ and $G(p)$ be Klein-bottle* Tonnetze*, and let F be a bijective mapping F: $F(G'(p')) = G(p)$; p' is a member of $G(p)$. F is an autography of $G'(p')$ onto $G(p)$ if $G' = G$.*

[4] KLEIN-BOTTLE *TONNEZTE* USING COMMUTATIVE GIS'S

[4.1] We will now construct a pitch-class Klein-bottle *Tonnetz* using operations from a commutative GIS.[37] Specifically, we will now give the intervals defined above as w, x, etc., as transpositions in the form T_n. Clearly, the T group, being cyclic, is commutative, as $T_nT_m = T_{m+n} = T_{n+m} = T_mT_n$. We require then two T_n operators, w and z, whose squares are equal; and, from this w and z, we obtain the generators w and $x = z^{-1}w$ of a Klein-bottle group G (2.8.2). One such example uses $w = T_2$ and $z = T_8$; each operation performed twice yields $y = T_4$. G may be generated, then, by $w = T_2$ and $x = z^{-1}w = T_4T_2 = T_6$.[38] Thus, $G = \langle\langle T_2\rangle,\langle T_6\rangle\rangle = \{T_0,T_2,T_4,T_6,T_8,T_{10}\}$.

[4.2] Figure 18 shows a portion of an abstract pitch-class Klein-bottle *Tonnetz* which is generated by $w = T_m$ and $x = T_6$, and which uses p as a pitch-class point of origin. Moreover, by 2.6.5, $z = xw = T_6T_m = T_{m+6}$. As such, $z^2 = T_{2(m+6)} = T_{2m} = w^2$, via 2.5.1.

Figure 18. A portion of an abstract pitch-class *Tonnetz* generated by w and x in a commutative G

```
  2m+p  =  2(m+6)+p
       / \
      /   \
   m+p___m+6+p
   / \     / \
  /   \   /   \
-6+p____p_____6+p
  \    / \    /
   \  /   \  /
  -m+p___-(m+6)+p
       \   /
        \ /
 -2m+p  =  -2(m+6)+p
```

[37] Henceforth, all pitch-class operations will be performed mod 12.

[38] Furthermore, following 2.11.1, all x's in commutative Klein-bottle *Tonnetze* generate involutions. Accordingly, x in this example is T_6, the only T_n operator with that property.

Recalling Figure 9, Figure 19 uses members of this *Tonnetz* to demonstrate the fundamental region of the Klein-bottle group.

Figure 19. The fundamental region of G in Figure 18

```
             (T_6_)
         m+p-->-(m+6)+p  =  (6-m)+p
           \          \
  (T_m_)  ^           ^  (T_m)
             \          \
              p---<--6+p
                  (T_6_)
```

Both T_m's in Figure 19 operate from bottom to top. However, the upper T_6 operates from left to right, whereas the lower T_6 operates from right to left,[39] providing the necessary identifications of sides.

[4.3] Before discussing pitch-class node content, we observe a few features of Figure 18 and its group of operations. First, as we noted in [3.6], all triangles in the figure possess the same GIS-interval content, $\{T_0,T_0,T_m,T_{-m},T_{-(6+m)},T_{6+m}\}$. Any triangle with this particular GIS-interval content forms a member of the same T/I set-class,[40] and also relates by either GIS-transposition or GIS-inversion to every triangle in the figure.[41] All triangles pointing in the same direction (t+ or t-) are transpositionally related to each other as pcsets, and are also GIS-transpositionally related to each other as networks. Triangles pointing in the opposite directions are inversionally related as pcsets and networks.

[4.4] We will now examine issues of node content. Definition 3.1.2 determines the number of distinct nodes in a T-only *Tonnetz*. In fact, since the T group, as a cyclic group, is simply transitive,[42] the earlier formula given in 2.10.2 suffices. Then, as $|X| = 2$, we need only determine whether or not $|W \cap X| > 1$ to arrive at $|G|$.

> THEOREM 4.4.1 *Given $w = T_m$ and $x = T_6$ (by 2.11.1), $X \subseteq W$ if there exists some integer i, such that $mi = 6$.*

[4.5] The number of distinct P-sets on which any one T-only G acts is precisely the index of G in the full T group. We call this set of P-sets a P_G set.

> DEFINITION 4.5.1 *$P_G = \{G(p) \mid p$ is a pitch-class integer$\}$.*

---

[39] Of course, being an involution, x in the graph of a commutative G may be said to operate both right-to-left and left-to-right. Thus, as we stated in Note 6, this fundamental region actually degenerates into a torus. However, since it satisfies the relations for the Klein-bottle group (2.8.2), we include it in the general category of "Klein-bottle *Tonnetze*."

[40] However, as we will see below, such networks do not necessarily contain all the trichords which belong to a particular set-class.

[41] Lewin (1997) discusses GIS-transposition and inversion.

[42] Lewin (1987, 157) defines the simply transitive property as follows: "Given any elements s and t of S, then there exists a unique member OP of STRANS such that OP(s) = t." We note that, in contrast to the T-only group, the full T/I group is not simply transitive (when it is acting on pcs, not pcsets).

The order of P_G for T-only G's is given as follows:

DEFINITION 4.5.2 $|P_G| = 12 / |G|$ *(for all commutative G's).*

In other words, we may generate the members of P_G by applying to p the right cosets of G in the full T group (or left cosets, as G is commutative).

[4.6] We return now to our example from [4.1], in which $w = T_2$, $x = T_6$, and $z = T_8$. Figure 20 shows a Klein-bottle *Tonnetz* that uses this G and p = 0 as point of origin.

Figure 20. A pitch-class *Tonnetz* generated by $w = T_2$ and $x = T_6$, using p = 0

```
    4______10____(4)
   / \    / \    /
  /   \  /   \  /
 8______2_____(8)
  \    / \    / \
   \  /   \  /   \
    0______6_____(0)
   / \    / \    /
  /   \  /   \  /
(10_____4_____(10))
```

In this *Tonnetz*, all triangles possess the GIS-interval content $\{T_0,T_6,T_2,T_{10},T_4,T_8\}$, and are accordingly members of set-class 3-8[026].

[4.7] As p varies among the twelve pitch-classes, we discover one other P-set on which the operations of this G may act: {1,3,5,7,9,11}, the set of odd pitch-classes. Figure 21 shows the network that results from G(p), where p = 1. Like Figure 20, it consists of a tessellation of triangles that belong to set-class 3-8[026]. Moreover, each triangle possesses the same GIS-interval content as those of Figure 20.

Figure 21. A pitch-class *Tonnetz* generated by $w = T_2$ and $x = T_6$, using p = 1

```
    5______11____(5)
   / \    / \    /
  /   \  /   \  /
 9______3_____(9)
  \    / \    / \
   \  /   \  /   \
    1______7_____(1)
   / \    / \    /
  /   \  /   \  /
(11_____5_____(11))
```

[4.8] In both Figures 20 and 21, G contains six members: the transposition operators with even indices. Therefore, by 4.5.2, we find here 12/6 = 2 distinct P-sets, the even and odd pitch-classes. Hence, for this G, $P_G = \{(0,2,4,6,8,10),(1,3,5,7,9,11)\}$.

[4.9] The beginning of no. 8, “Gargoyles,” from George Crumb’s *Makrokosmos*, Vol. II, presents us with an analytical example. The piece opens with a series of 3-5[016] trichords in both hands, the left hand’s pcsets being inversionally related to those of the right hand. (See Example 3.) These pitch-class sets may be plotted as subnetworks on a T-only Klein-bottle *Tonnetz*. The entire network’s underlying graph may be constructed using w = T_5 and z = T_{11}; then, $w^2 = z^2 = T_{10}$. Using 2.8.2, we give the generators of this G as w = T_2, and x = $z^{-1}w = T_6$. Then, application of this group of operators to pitch-class 3 obtains the Klein-bottle *Tonnetz* shown in Figure 22.

Example 3. Crumb, no. 8, "Gargoyles," from *Makrocosmos*, Vol. II, mm. 1-3

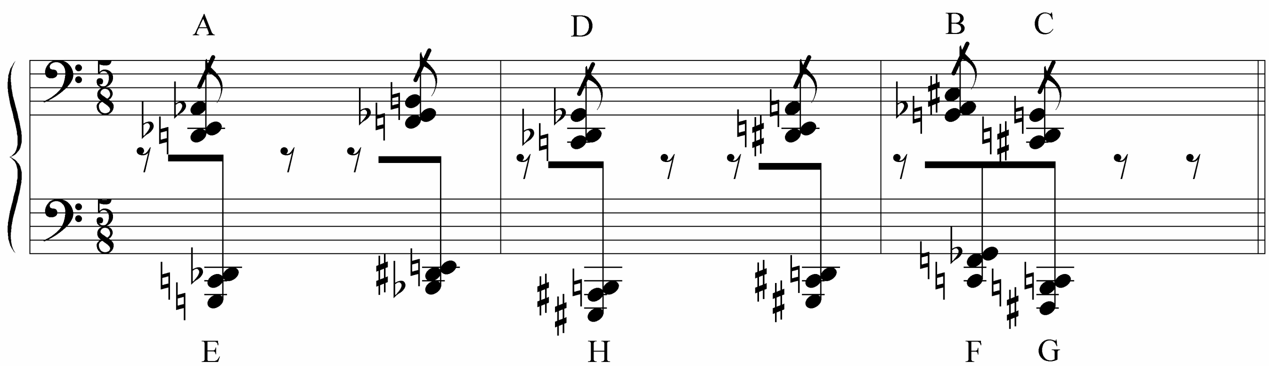


Figure 22. Klein-bottle *Tonnetz* showing 3-5[016] trichords in the opening of “Gargoyles”

```
  3---9---3
 / \ / \ /
10--4---10
 \ / \ /H\
  5---11--5
 /F\ /G\ /(F)
0---6---0
 \ / \D/E\
  7---1---7
 / \B/ \C/
2---8---2
 \ / \A/ \
  9---3---9
```

[4.10] Unlike the earlier Schoenberg example (Example 1a), the inversion operators present in this example are not part of the passage’s overall growth process. Consequently, we do not require I_n members in G to model our interpretation of the music. Rather, the inversionally related trichords between the hands move in parallel motion, suggesting a type of contextual inversion operation that is inherent in the upward- and downward-pointing triangles of the figure.[43] As we pointed out in [4.3], a T-only Klein-bottle *Tonnetz* includes inverted, as well as non-inverted, forms of a pcset.

[43] By graphic analogy to the Oettingen/Riemann *Tonnetz*, the trichords in the two hands of each gesture in the Crumb example relate by the neo-Riemannian operator PLP, which, on the former network, yields the hexatonic pole.

[4.11] As a result, in addition to pitch-classes, we may also model operations among Example 3's pitch-class sets on Figure 22. For purposes of illustration, we will consider only those trichords labeled A-H in the example. (These trichords are plotted on the figure.) Considering first only the right hand, we see that trichord B obtains by applying $w = T_5$ to trichord A; accordingly, we may perform the same glide reflection to the triangle marked A to reach B, as we did to pitch-class 3 to reach 8. Similarly, trichord C obtains from A via $z = T_{11}$, which also carries pitch-class 3 to 2. Moreover, both glide reflections performed twice, $w^2 = z^2 = y$, carry trichord A to D, or pitch-class 3 to 1. Furthermore, the trichords of the left hand are related in precisely the same way. Trichord E moves to F via w, to G via z, and to H via y.

[4.12] We need not limit ourselves to trichords, and indeed, a more natural way of looking at the passage might be in terms of the 6-38[012378] hexachords formed by taking the two hands of each gesture together. Figure 23 shows these hexachords as parallelograms, incorporating the two trichords of each gesture. We may easily see the two glide reflections which carry the A/E parallelogram to B/F and C/G, both of which squared yield a translation to D/H.

Figure 23. Hexachords in the opening of "Gargoyles"

```
            10
           /H\
  11  5   11  5
 /G\ /F\ /   /
6   0   6   0
 \   \   \D/E\
  1   7   1   7
   \C/ \B/   /
    2   8   2
        \A/
         3
```

[5] KLEIN-BOTTLE *TONNEZTE* USING NON-COMMUTATIVE GIS'S

[5.1] In [3], we constructed an abstract Klein-bottle *Tonnetz* using intervals from a generalized, non-commutative GIS. Now we are ready to consider the pitch-class representations of these types of networks, using operators from the traditional, non-commutative T/I group.[44] We will also present an analytical example that uses a non-commutative, generalized quaternion T/M subgroup.

[5.2] As before, we require the relation $w^2 = z^2$ (2.5.1) to arrive at the proper graphic identifications for a Klein-bottle's fundamental region. Furthermore, using the dihedral T/I group, all triangles formed by mutually adjacent nodes in the graphs of such G's must incorporate two I_n operators and one T_n operator. Otherwise, Condition (A) of Lewin's (1987, 26) definition of a GIS, which guarantees the transitive property, is not met, pace O'Donnell's (1998, 56-60) concepts of well-formed and practical K-nets as applied to trichords.

[5.3] We find two categories of these *Tonnetze*. The first includes those in which w and z are both I_n operators, and x is a T_n operator. The second includes those in which x and either w or z, but not

[44] In this section, we will also sometimes refer to the T/I group using the notation TTO_{24}, following Morris (1987).

both, are I_n operators, and the other is a T_n operator.[45]

DEFINITION 5.3.1 *Category 1: x is of the form T_n.*

DEFINITION 5.3.2 *Category 2: x is of the form I_n.*

In either event, the orders of W and Z are always equal, and both are involutions.

THEOREM 5.3.3 *In any Klein-bottle* Tonnetz *which incorporates both T_n and I_n operators,* $|W| = |Z| = 2$.

[5.4] Figures 24a-c show portions of abstract pitch-class *Tonnetze* that use T_n and I_n operators, and their fundamental regions. Figure 24a models Category 1,[46] and Figures 24b-c show the two possibilities for Category 2.

Figure 24a. A portion of an abstract T/I *Tonnetz* using $w = I_m$ and $x = T_n$, and its fundamental region (Category 1)

```
a)        p               b)       T_n_
         / \                  (m-p)-->--((n+m)-p)
        /   \                      \        \
     m-p_____(n+m)-p          I_m_ ^         ^ I_2n+m_
       / \   / \                    \        \
      /   \ /   \                   (p)--<---(n+p)
  -n+p______p______n+p                  T_-n_
      \   / \   /
       \ /   \ /
     m-p_____(n+m)-p
        \   /
         \ /
          p
```

Figure 24b. A portion of an abstract T/I *Tonnetz* using $w = I_m$ and $x = I_n$, and its fundamental region (Category 2)

```
a)        p               b)       I_n_
         / \                  (m-p)-->--(6+p)
        /   \                      \        \
     m-p_____6+p              I_m_ ^         ^ I_m_
       / \   / \                    \        \
      /   \ /   \                   (p)--<---(n-p)
   n-p______p______n-p                  I_n_
      \   / \   /
       \ /   \ /
     m-p_____6+p
        \   /
         \ /
          p
```

[45] We note that all G's in Category 2, as well as all order 4 G's in Category 1, are, in fact, commutative. However, since they consist of both T_n and I_n operators, which do not normally commute, we include them in this section.

[46] In Figure 24a, using the identifications of the sides from [2.7], the edge (interval) represented by (p,m-p) maps onto that of (n+p,(n+m)-p) via a conjugation of w by x (i.e., $T_n(I_m)T_{-n} = I_{2n+m}$), and (m-p,(n+m)-p) maps onto (n+p,p) via a conjugation of x by z^{-1} (i.e., $I_{n+m}(T_n)I_{n+m} = T_{-n}$). Similar identifications may be made for Figures 24b–c.

Figure 24c. A portion of an abstract T/I *Tonnetz* using w = T_6 and x = I_n, and its fundamental region (Category 2)

```
a)        p
         / \
        /   \
     6+p_____(n-6)-p
      / \    / \
     /   \  /   \
 n-p______p______n-p
     \   / \    /
      \ /   \  /
     6+p_____(n-6)-p
        \    /
         \  /
          p
```

```
b)       I_n_
   (6+p)-->--((n-6)-p)
        \         \
   T_6_ ^          ^ T_6_
          \         \
          (p)--<---(n-p)
               I_n_
```

[5.5] Taken as Klumpenhouwer networks, all triangles in any such graph are isographic to each other. First, as a consequence of W's being an involution, the interval series of any t+ and t- triangles that share a common *Einheit* are equivalent. Therefore, we need only consider here one or the other; we will arbitrarily choose t+ triangles. Now, in Category 1 *Tonnetze*, all t+/e triangles are positively isographic to each other, since, for any two t+/e triangles whose *Einheiten* are T_{nj}(p) and $T_{n(j+k)}$(p), 3.5.1 gives respectively the following interval series: (T_n, $I_{2nj+2n+m}$, $I_{2nj+2n-n+m}$) and (T_n, $I_{2nj+2n+m+2nk}$, $I_{2nj+2n-n+m+2nk}$). The T_n operators are equivalent in the two series, and both I_n operators of the second are 2nk greater than those of the first. Hence, the latter triangle is isographic to the former via the outer automorphism $\langle T_{2nk}\rangle$ of the T/I group, or the inner automorphism [T_{nk}]. Furthermore, all t+/e and t+/o triangles are negatively isographic to each other. For example, a t+/e triangle with the *Einheit* T_{nj}(p) and interval series (T_n, $I_{2nj+2n+m}$, $I_{2nj+2n-n+m}$), and a t+/o triangle with *Einheit* $I_{m+n(j+k)}$(p) and interval series (T_{-n}, $I_{2nj+2nk-2n+m}$, $I_{2nj+2nk-n+m}$) (by 3.5.2) are isographic via the outer automorphism $\langle I_{2(nk+m)}\rangle$, or inner automorphism [I_{nk+m}]. We could work similar calculations for Category 2, but as $|W| = |X| = 2$ in all these *Tonnetze*, their respective constituent triangles all possess the same interval series, as we saw in [3.7].

[5.6] Let us now proceed to issues of node content. We determine the number of distinct pitch-class nodes in P by using 3.1.2. In T/I Klein-bottle *Tonnetze*, two possibilities for $|W(p)|$ exist: 1 or 2. We give here the circumstances in which $|W(p) \cap X(p)| = 2$, starting with Category 1.

> THEOREM 5.6.1 *Given* $w = I_m$, *and* $x = T_n$ *(Category 1),* $W(p) \subseteq X(p)$ *if there exists some integer i, such that* $ni+p = m-p$.

So, $|W(p) \cap X(p)| > 1$ only when $W(p) \subseteq X(p)$, and $|W(p)| = 2$.

[5.7] Next, we consider examples of $|W(p) \cap X(p)| > 1$ in Category 2 *Tonnetze*, recalling that, in these networks, either x or z is T_6 (by 5.3.3). We may eliminate all examples in which z = T_6.

> THEOREM 5.7.1 *If* $z = T_6$ *(Category 2), then* $|W(p) \cap X(p)| = 1$.

Therefore, for $|W(p) \cap X(p)|$ to be greater than 1 in Category 2 *Tonnetze*, w must be T_6. Now, by using the same argument as 5.6.1, but exchanging the T_n and I_n forms of w and x, we observe the situation in which $X(p) \subseteq W(p)$.

COROLLARY 5.7.2 *Given $w = T_n$ and $x = I_m$ (Category 2), $X(p) \subseteq W(p)$ if there exists some integer i, such that $ni+p = m-p$.*

Thus, $|W(p) \cap X(p)| > 1$ in Category 2 only when $X(p) \subseteq W(p)$, and $|X(p)| = 2$. The remarkable conclusion of 5.6.1 and 5.7.1-2 is that the same G may produce P-sets of different cardinalities.

[5.8] From the above discussion, it is implied that the number of distinct P-sets on which the same G may act is not necessarily given by the index of G in the T/I group, as it was for the simply transitive examples in [4]. Let us now examine the situations in which we find degenerate P-sets under right cosets of G. First, we start with a few definitions.

DEFINITION 5.8.1 *$P_G = \{G(p) \mid p$ is a pitch-class integer$\}$ (same as 4.5.1).*

P_G is the set of all distinct P-sets on which a particular G may act. R_G, then, is the set of all right cosets of that G in the T/I group.

DEFINITION 5.8.2 *$R_G = \{Gc_i \mid c$ is a member of TTO_{24}, and i is an integer mod $[TTO_{24} : G]\}$.*

Using the members of R_G applied to p, we give the resulting set of degenerate P-sets as D_G.

DEFINITION 5.8.3 *$D_G = \{Gc_i(p) \mid Gc_i(p) = Gc_j(p)$, and i is not equal to j; Gc_i, Gc_j are members of $R_G\}$.*

Then, the cardinality of P_G is the difference of the order of D_G from the index of G in the T/I group.

DEFINITION 5.8.4 *$|P_G| = [TTO_{24} : G] - |D_G|$.*

[5.9] For two distinct cosets to produce degenerate P-sets, certain relations have to be met. These situations are given below in the form of lemmata; the interested reader may work their proofs, which are eliminated here for reasons of space. We start with Category 1, in which $w = I_m$ and $x = T_n$.

LEMMA 5.9.1 *If $Gc_i = G(T_x)$, and $Gc_j = G(T_{x'})$, then $Gc_i(p) = Gc_j(p)$ if there exists an integer i, such that $(ni+x)+p = x'+p$, and/or $((m-ni)-x)-p = x'+p$.*

LEMMA 5.9.2 *If $Gc_i = G(T_x)$, and $Gc_j = G(I_{y'})$, then $Gc_i(p) = Gc_j(p)$ if there exists an integer i, such that $(ni+x)+p = y'-p$, and/or $((m-ni)-x)-p = y'-p$.*

LEMMA 5.9.3 *If $Gc_i = G(I_y)$, and $Gc_j = G(I_{y'})$, then $Gc_i(p) = Gc_j(p)$ if there exists an integer i, such that $(ni+y)-p = y'-p$, and/or $((m-ni)-y)+p = y'-p$.*

LEMMA 5.9.4 *If $Gc_i = G(I_y)$, and $Gc_j = G(T_{x'})$, then $Gc_i(p) = Gc_j(p)$ if there exists an integer i, such that $(ni+y)-p = x'+p$, and/or $((m-ni)-y)+p = x'+p$.*

For Category 2, we may similarly determine the degenerate P-sets by exchanging the T_n and I_n forms of w and x from Category 1. In conclusion, we observe that, using G's with both T_n and I_n operators, not only may the same G act on P-sets of different sizes, but also the number of these P-sets is not necessarily the index of G in the full T/I group.

[5.10] Let us examine one such Klein-bottle *Tonnetz*, that which uses w = I_7, x = T_3, and p = 3 (Category 1). (See Figure 25.)

Figure 25. The network of w = I_7, x = T_3, and p = 3

```
  (3---6---9---0--(3))
  / \ / \ / \ / \ /
 4---7---10--1--(4)
  \ / \ / \ / \ / \
   3---6---9---0--(3)
```

This network is essentially the same as the one presented in [1.4] (Figure 2) to model trichords in the beginning of "Nacht" from *Pierrot Lunaire*, when we alluded to its being a Klein-bottle *Tonnetz*.

[5.11] We note that all triangles in Figure 25 belong to the same set-class, 3-3[014]; however, not all triangles possess the same GIS-interval content. Only triangles related by some member in the center of G (3.6.1), or which share an *Einheit* with these triangles (3.5.6), have the same interval series. The center of this particular G is $\langle T_6 \rangle$. Therefore, in the Schoenberg example, trichords that are in corresponding positions within the canonic framework (canon at T_6) have the same GIS-interval content.

[5.12] To generate a related *Tonnetz*, using the same G, in which all triangles possess the same interval series, and therefore the same GIS-interval content, we permute the nodes of G(p) according to the anti-homomorphism in 3.9.1; we label the resulting network H(p). Figure 26 shows this action on Figure 25.

Figure 26. H(p), where w = I_7, x = T_3, and p = 3

```
  (3-->6-->9-->0->(3))
  / \ / \ / \ / \ /
 4<--1<--10<-7<--(4)
  \ / \ / \ / \ / \
   3-->6-->9-->0->(3)
```

Now, all triangles do indeed have the same interval series (T_3,I_4,I_7), hence the same GIS-interval content. However, only those triangles related by some member in the center of G, or which share an *Einheit* with those triangles, belong to the same set-class. Otherwise, all triangles are GISZ-related.[47] In fact, this *Tonnetz* demonstrates clearly the topology of the Klein bottle. Note, in particular, that all parallel diagonal edges in any one row represent the same I_n operator, and the direction of T_n arrows reverses in alternate rows.

[47] Lewin (1997) discusses the GISZ-relation in the context of commutative and non-commutative GIS's.

[5.13] As Figure 25's G is of order 8, it has 24/8 = 3 right cosets in the T/I group; in other words, $[TTO_{24} : G] = 3$. Figures 27a-c show the three networks that derive from applying these cosets to p = 3.

Figure 27. The three right cosets Gc of G applied to p = 3

```
a) c is a member of   b) c is a member of   c) c is a member of

T_0_,T_3_,T_6_,T_9_,  T_1_,T_4_,T_7_,T_10_  T_2_,T_5_,T_8_,T_11_
I_1_,I_4_,I_7_,I_10_  I_0_,I_3_,I_6_,I_9_   I_2_,I_5_,I_8_,I_11_


 (3---6---9---0--(3))   (4---7---10--1--(4))
 / \ / \ / \ / \ /      / \ / \ / \ / \ /
4---7---10--1--(4)     3---6---9---0--(3)    (2---5---8---11-(2)
 \ / \ / \ / \ / \      \ / \ / \ / \ / \      \ / \ / \ / \ / \
  3---6---9---0--(3)     4---7---10--1--(4)     5---8---11--2--(5)
```

The P-sets of Figures 27a and b are equivalent, hence $|D_G| = 1$ (5.8.3). Therefore, in this case, $|P_G| = [TTO_{24} : G] - |D_G| = 2$ (5.8.4).

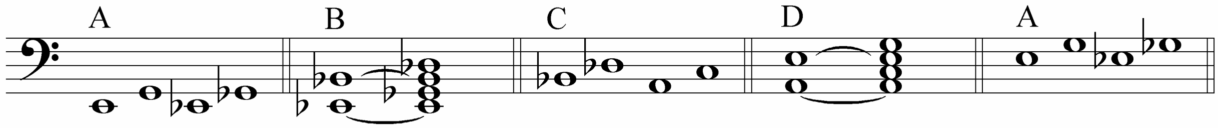
[5.14] In [1.4], our analysis of the opening canon in "Nacht" (Example 1a) fails to address two important features of the passage. First, it does not address the relationship between set-classes 3-3[014] and 3-11[037]. However, the excerpt's melodic material derives largely from the former, while the harmonic material uses the latter. Second, it does not address the presence of the pitch-class 8 in m. 3, the one note outside the prevailing octatonic collection. Both these issues may be explored in terms of a T/M Klein-bottle *Tonnetz*.

[5.15] Figure 28 shows one such network, generated here by $w = T_7M$ and $x = T_4$; then, $z = xw = T_{11}M$. In the figure, we plot various tetrachords from the excerpt, labeled A though D (see Example 4).

Figure 28. Network of melodic and harmonic tetrachords in the opening canon of "Nacht"

```
(6)  10  2   6   10
 |
 | \
 |  \
 3 A 7   11  3   7
  \  |          /|
   \ |         / |
    \|        /  |
 0   4   8   0 D 4
 |\          |  /
 | \         | /
 |  \        |/
 9 C 1   5   9   1
  \  |          /|
   \ |         / |
    \|        /  |
 6  10   2   6 B 10
 |\          |  /
 | \         | /
 |  \        |/
 3 A 7   11  3   7
  \  |
   \ |
    \|
 0   4   8   0   4
```

Example 4. Melodic and harmonic tetrachords in "Nacht," mm. 1-3

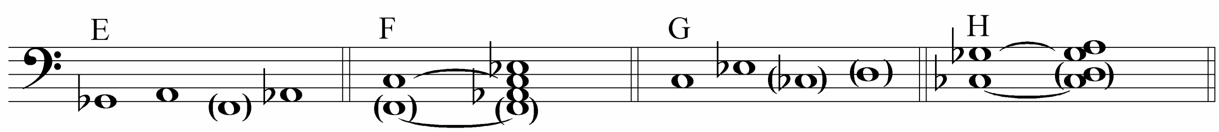


Subnetwork A presents the opening melodic gesture, {4,7,3,6}. On beat 3 of m. 1, this gesture restates canonically in the right hand with C = {10,2,9,1}, initiating a T_6-cycle (A C [A]). Each of these tetrachords forms a member of set-class 4-3[0134], which may be viewed as the union of two inversionally related 3-3[014] trichords.

[5.16] Each statement of the melodic motive overlaps with a subsequent one, giving us parallel perfect fifths, leading to a harmonic accumulation. The first of these harmonic arrivals is B = {3,10,6,1}. It is a member of set-class 4-26[0358], which consists of two inversionally related triads. The next accumulation is D = {9,4,1,7} in m. 2. These harmonic events also spell a T_6-cycle on subnetworks (B D [B]). Moreover, B is the T_7M-transform of A = {3,4,6,7}, and, accordingly, the latter's semitones transform into the former's perfect fifths. Subnetwork B, then, also relates to C via T_7M, C to D, and so forth. Therefore, we observe a complete T_7M-cycle, (A B C D [A]), in the example, shown here as a cycle of glide reflections.

[5.17] The above T_7M-cycle acts on the octatonic collection (0,1,3,4,6,7,9,10), which does not include the passage's final pitch-class, 8. Its presence here can be explained in terms of another T_nM-cycle, which is only implied in the excerpt. We may reveal an incomplete, hidden repetition of the melodic and harmonic motives, beginning on the G♭ downbeat of m. 2: E = {6,9,[5],8} and F = {[5],0,8,3}, respectively. (See Example 5, segments E and F.)

Example 5. Secondary cycle of melodic and harmonic tetrachords in "Nacht," mm. 1-3



Pitch-class 5 never appears in the passage; hence, its inclusion here is merely conjectured. The metric displacement of pitch-class 8 in m. 3, then, suggests a written-out ritardando, signaling the end of the phrase. Subnetwork E relates to F via $T_{11}M$, initiating another T_nM-cycle, (E,F,G,H). Example 5 shows a reconstruction of this entire cycle, if it had been fully realized in the music, and Figure 29 plots its tetrachords E-H on the same network as above. The product of this $T_{11}M$-cycle and the earlier T_7M-cycle forms a Klein-bottle-group K with which we may analyze the entire passage,[48] accounting particularly for those aspects not addressed in our earlier analysis.

[48] This particular group, K = $\{T_m, T_nM \mid m \text{ is even, } n \text{ is odd}\}$, is isomorphic to the generalized quaternion group of order 12, Q_{12} (see Note 17). Accordingly, it contains three non-commuting order 4 subgroups, $\langle T_1M\rangle = \langle T_7M\rangle$, $\langle T_3M\rangle = \langle T_9M\rangle$, and $\langle T_5M\rangle = \langle T_{11}M\rangle$, which intersect in the order 2 center of the group, $\langle T_6\rangle$.

Figure 29. T_{11}M-cycle in the reconstructed hidden repetition

```
  5  (9)  1   5   9
     /|
    / |
   /  |
 (2)H(6)  10 (2)   6
  |  /        |\
  | /         | \
  |/          |  \
(11)  3   7 (11)G 3
     /|         \  |
    / |          \ |
   /  |           \|
  8 F 0   4   8   0
  |  /        |\
  | /         | \
  |/          |  \
 (5)  9   1  (5)E 9
               \  |
                \ |
                 \|
  2   6   10  2   6
```

[5.18] Our new analysis of the passage can also address certain aspects of recursion in the excerpt. Specifically, we can find relations between the interpretation of certain prominent pitch-class sets in the excerpt and a supernetwork on subnetworks A-H above. Figure 30 shows one such pitch-class set from mm. 1-2: J = {3,6,7,10}, a member of 4-17[0347]. Like the members A, C, E, and G of set-class 4-3[0134] above, J is the union of two inversionally related 3-3[014] trichords. Similarly, like B, D, F, and H from 4-26[0358], J also contains both a major and a minor triad. The figure shows an interpretation of this pitch-class set, using operators from our Klein-bottle group.

Figure 30.

```
         a)                         b)

           T_8_                        [T_8_]
         6<----10                  E=G<----A=C
         ^      ^                      ^      ^
   T_3_M |  J   |T_11_M    [T_11_M]|   N   |[T_7_M]
         |      |                      |      |
         3<----7                   F=H<----B=D
           T_8_                        [T_8_]
```

Figure 30b, then, interprets subnetworks A-H. The correspondence of the two figures derives from an inner automorphism of our Klein-bottle group. Specifically, this automorphism is accomplished by a conjugation of K by T_4. In other words, $T_4(K)T_8$ gives us the following mapping of elements of K.

TABLE 5.18.1. Mapping of K onto $T_4(K)T_8$

```
Elements    Conjugation    Elements
of K        by T_4_        of K^T_4_^
----------------------------------
T_0_           -->         T_0_
T_2_           -->         T_2_
T_4_           -->         T_4_
T_6_           -->         T_6_
T_8_           -->         T_8_
T_10_          -->         T_10_
T_1_M          -->         T_9_M
T_3_M          -->         T_11_M
T_5_M          -->         T_1_M
T_7_M          -->         T_3_M
T_9_M          -->         T_5_M
T_11_M         -->         T_7_M
```

[5.19] Under conjugation by T_4, the operators of Figure 30a map one-to-one onto those that determine the hyper-operators of 30b. Furthermore, $(X^{T\_4})(Y^{T\_4}) = (XY)^{T\_4}$, for any X and Y in K. Therefore, the two groups—one of operators, and the other of hyper-operators—are related to each other, showing again a remarkable degree of consistency among the various levels of the excerpt. Pitch-class 8 is of particular interest in this analysis; while initially seeming aberrant, its presence ultimately made possible the recursion between the pitch-class-network level and the supernetwork level.

[6] K-NET ISOGRAPHIES AMONG KLEIN-BOTTLE *TONNETZE*

[6.1] We have already discussed briefly the network isographies among triangles formed by mutually adjacent nodes in Klein-bottle *Tonnetze*. Now we turn our attention to the notion of entire T/I Klein-bottle *Tonnetze* as Klumpenhouwer networks, and the various isographies which relate them to each other.

[6.2] Network isographies obtain from automorphic mappings of a group of operations onto itself. Of particular relevance here are the T and T/I groups, as they are the most studied in the literature on Klumpenhouwer networks.[49] We find four categories of these automorphisms: $\langle T_j \rangle$, $\langle M_j \rangle$, $\langle MI_j \rangle$, and $\langle I_j \rangle$.[50]

DEFINITION 6.2.1 $\langle T_j \rangle T_n = T_n$; $\langle T_j \rangle I_n = I_{n+j}$.

[49] Lewin (1990, Appendix B) describes a system for including T_nM and T_nMI operations in Klumpenhouwer networks, including a subgroup, RECURSE, of Aut(TTO_{48}) that models such networks' isographies recursively. However, in this section, we will not pursue M and MI operations further than their implied presence in Aut(TTO_{24}).

[50] Within this section, angle brackets surrounding a twelve-tone operator do not signify a cycle of this operator. Rather, following Klumpenhouwer's (1998, 88) notation, they indicate some outer automorphism of the appropriate group of operations. The four types of these automorphisms of the T/I group are defined using a different notation in Lewin (1990, 88). Lewin's $F\langle u,j \rangle$ function maps the group according to the following scheme: under $F\langle u,j \rangle$, T_n operators map onto T_{un} operators, and I_n operators map onto I_{un+j} operators. The only values for u which obtain automorphisms of the T/I group are 1, 5, 7, and 11: the order 12 coprimes in $\mathbf{Z}_{12}$. Here, we indicate Lewin's $F\langle 1,j \rangle$ as $\langle T_j \rangle$, $F\langle 5,j \rangle$ as $\langle M_j \rangle$, $F\langle 7,j \rangle$ as $\langle MI_j \rangle$, and $F\langle 11,j \rangle$ as $\langle I_j \rangle$.

DEFINITION 6.2.2 $\langle M_j \rangle T_n = T_{5n}$; $\langle M_j \rangle I_n = I_{5n+j}$.

DEFINITION 6.2.3 $\langle MI_j \rangle T_n = T_{7n}$; $\langle MI_j \rangle I_n = I_{7n+j}$.

DEFINITION 6.2.4 $\langle I_j \rangle T_n = T_{11n}$; $\langle I_j \rangle I_n = I_{11n+j}$.

We refer to these automorphisms as hyper-operators, which may form supernetworks. We also say that the hyper-operator $\langle T_j \rangle$ is determined by the TTO T_j, or that T_j is the determinant of $\langle T_j \rangle$; and so forth with $\langle M_j \rangle$ and M_j, $\langle MI_j \rangle$ and MI_j, and $\langle I_j \rangle$ and I_j.

[6.3] Concerning G's in the commutative T group, these labels refer to the following mappings, regardless of the value of j: $\langle T_j \rangle T_n = T_n$, $\langle M_j \rangle T_n = T_{5n}$, $\langle MI_j \rangle T_n = T_{7n}$, and $\langle I_j \rangle T_n = T_{11n}$. In other words, two T Klein-bottle *Tonnetze* are isographic to each other if all T_n intervals between nodes in one relate to those of the other by multiplication by 1, 5, 7, or 11. We will not pursue a detailed study of GA-isography in T Klein-bottle *Tonnetze*. For our purposes, it does not lead to a satisfying result, as we cannot reveal a convincing recursive supernetwork of networks. Clearly, as any $\langle T_j \rangle$ hyper-operator (6.2.1) maps some T_n member of G onto itself, all $\langle T_j \rangle$'s are merely identity mappings in this context. Therefore, they do not form analogous structures between nodes and networks-as-nodes.

[6.4] In contrast, GA-isographies among T/I Klein-bottle *Tonnetze* are of interest. The $\langle T_j \rangle$ and $\langle I_j \rangle$ hyper-operators have fully corresponding analogues in G's T_n and I_n operators. In fact, as we will now demonstrate, any T/I G determines a set of networks that forms at least one supernetwork whose group of hyper-operators is isomorphic to G itself. We will now investigate that situation.

[6.5] Specifically, each T/I G forms as many distinct networks as it has pairs of generators (w,x). For Category 1 *Tonnetze*, this number is equal to the number of automorphic mappings of G onto itself, via the forty-eight hyper-operators. (We will address Category 2 G's below in Note 51.) We now define the set HYP of all hyper-operators that map G onto itself.

DEFINITION 6.5.1 $HYP = \{\langle h \rangle \mid \langle h \rangle G = G\}$.

Because these automorphisms map not just G, but the entire T/I group onto itself, certain different members of HYP may induce equivalent mappings of a particular G onto itself. For example, let G = $\{T_0,T_4,T_8,I_1,I_5,I_9\}$. Table 6.5.2 shows the elementwise automorphic mapping of this G onto itself under $\langle I_2 \rangle$.

TABLE 6.5.2 Mapping of $\langle I_2 \rangle$G onto G

```
Member           Member
of G   <I_2_>    of G
---------------------
T_0_   <-->      T_0_
T_4_   <-->      T_8_
T_8_   <-->      T_4_
I_1_   <-->      I_1_
I_5_   <-->      I_9_
I_9_   <-->      I_5_
```

Table 6.5.3 shows the same mapping under $\langle M_8 \rangle$.

TABLE 6.5.3 Mapping of $\langle M_8 \rangle$G onto G

Member of G	<M_8_>	Member of G
T_0_	<-->	T_0_
T_4_	<-->	T_8_
T_8_	<-->	T_4_
I_1_	<-->	I_1_
I_5_	<-->	I_9_
I_9_	<-->	I_5_

In terms of G alone, these two mappings are obviously the same. In total, for this G, we find only six distinct mappings: (1) $\langle T_0 \rangle = \langle MI_6 \rangle$, (2) $\langle T_4 \rangle = \langle MI_{10} \rangle$, (3) $\langle T_8 \rangle = \langle MI_2 \rangle$, (4) $\langle I_2 \rangle = \langle M_8 \rangle$, (5) $\langle I_6 \rangle = \langle M_0 \rangle$, $\langle I_{10} \rangle = \langle M_4 \rangle$. We will call the set of distinct mappings of any G onto itself MAP.

> DEFINITION 6.5.4 *MAP = {F | F is an automorphic mapping of G onto G, wherein T_n members of G map onto T_n members, and I_n members map onto I_n members}.*

Thus, |MAP| ≤ |HYP|.

[6.6] On account of our pursuit of only recursive structures, we will henceforth limit ourselves to group automorphisms via $\langle T_j \rangle$ and $\langle I_j \rangle$. Therefore, we give a subset of HYP, TIHYP, whose members are only in the forms $\langle T_j \rangle$ and $\langle I_j \rangle$.

> DEFINITION 6.6.1 *TIHYP = {$\langle h \rangle$ | h is a member of TTO_{24}, and $\langle h \rangle$G = G}.*

As in HYP, certain members of TIHYP represent the same mapping. Therefore, we define one further set, TIMAP, which consists of the distinct mappings of a particular G onto G in TIHYP.

> DEFINITION 6.6.2 *TIMAP = {F | the distinct automorphic mappings of a particular G onto G in TIHYP}.*

Accordingly, |TIMAP| ≤ |TIHYP|.

[6.7] Now, each network which expresses the same G may serve as a node in a supernetwork whose edges represent $\langle T_j \rangle$ or $\langle I_j \rangle$ intervals. Furthermore, the groups of these supernetworks are isomorphic to G itself. For instance, consider the eight networks that express G = $\{T_0,T_3,T_6,T_9,I_1,I_4,I_7,I_{10}\}$. Table 6.7.1 lists these eight networks by (w,x) pair, and assigns them a label g1 through g8.

TABLE 6.7.1 The eight networks that express G = $\{T_0,T_3,T_6,T_9,I_1,I_4,I_7,I_{10}\}$

Label	(w,x) pair
g1	(I_1_,T_9_)
g2	(I_4_,T_3_)
g3	(I_1_,T_3_)
g4	(I_10_,T_9_)
g5	(I_4_,T_9_)
g6	(I_7_,T_9_)
g7	(I_7_,T_3_)
g8	(I_10_,T_3_)

[6.8] Figure 31a replicates one of these networks, g8; 31b details its fundamental region.

Figure 31. The network of $w = I_{10}$, $x = T_3$, using $p = 0$, with fundamental region

```
a)

   (0-----3-----6-----9----(0))
   / \   / \   / \   / \   /
  /   \ /   \ /   \ /   \ /
 10----1-----4-----7----(10)
  \   / \   / \   / \   / \
   \ /   \ /   \ /   \ /   \
    0-----3-----6-----9----(0)
```

```
b)
      T_3_
 10---->-----1
  \         / \
   \       /   \
  I_10_  I_1_  I_4_
     \   /       \
      \ /         \
       0-----<-----3
           T_9_
```

This G maps automorphically onto itself under eight $\langle T_j \rangle$ and $\langle I_j \rangle$ hyper-operators. Using the group TIHYP = $\{\langle T_0\rangle,\langle T_3\rangle,\langle T_6\rangle,\langle T_9\rangle,\langle I_2\rangle,\langle I_5\rangle,\langle I_8\rangle,\langle I_{11}\rangle\}$, Figure 32a places g8 into a supernetwork; Figure 32b, again, shows its fundamental region.

Figure 32. The supernetwork of $w = \langle I_{11}\rangle$, $x = \langle T_3\rangle$, and $p = g8$, with its fundamental region

```
a)

   (g8----g3----g2----g6---(g8))
   / \   / \   / \   / \   /
  /   \ /   \ /   \ /   \ /
 g1----g5----g7----g4---(g1)
  \   / \   / \   / \   / \
   \ /   \ /   \ /   \ /   \
    g8----g3----g2----g6---(g8)
```

```
b)
     <T_3_>
 g1---->-----g5
  \         / \
   \       /   \
<I_11_> <I_2_> <I_5_>
     \   /       \
      \ /         \
       g8----<-----g3
          <T_9_>
```

This group of hyper-operators is isomorphic to Figure 31's group of operators. Table 6.8.1 gives the mapping F of the isography between Figures 31 and 32.

TABLE 6.8.1 Mapping of F(G) onto TIHYP

Member of G	F	Member of TIHYP
T_0_	<--->	<T_0_>
T_3_	<--->	<T_3_>
T_6_	<--->	<T_6_>
T_9_	<--->	<T_9_>
I_1_	<--->	<I_2_>
I_4_	<--->	<I_5_>
I_7_	<--->	<I_8_>
I_10_	<--->	<I_11_>

[6.9] The networks of order 12 and order 6 T/I G's fit similarly into isographic supernetworks. However, the order 24 and order 4 G's are somewhat exceptional. The sole order 24 G, TTO_{24}, may be expressed by forty-eight distinct networks. Any one of these networks maps onto each of the forty-eight, using a different hyper-operator for each one. Accordingly, these mappings exhaust all forty-eight $\langle T_j\rangle$, $\langle M_j\rangle$, $\langle MI_j\rangle$, and $\langle I_j\rangle$ hyper-operators. However, since we choose not to consider $\langle M_j\rangle$ and $\langle MI_j\rangle$ here, we are left with two supernetworks of order 24 whose groups of hyper-operators map onto themselves via any $\langle T_j\rangle$ or $\langle I_j\rangle$. In the constituent G's of one of these sets of networks, x is always either T_1 or T_{11}, and in the other set of networks, it is T_5 or T_7. The group of hyper-operators for either supernetwork is isomorphic to G = TTO_{24}.

[6.10] The networks of order 4 are also exceptional. In Category 1,[51] we find only two networks for each G. This situation exists simply because, in all cases, x = T_6, and is accordingly its own inverse. Hence, any mapping under $\langle I_j\rangle$ is structurally identical to some mapping under $\langle T_j\rangle$. Thus, 2|TIMAP| = |TIHYP|. We may still fashion an order 4 supernetwork, but it will possess two degenerate networks-as-nodes. (See Table 6.10.1 and Figure 33 for one such example.)

TABLE 6.10.1 The two networks which express G = $\{T_0,T_6,I_2,I_8\}$

Label	(w,x) pair
g1	(I_2_,T_6_)
g2	(I_8_,T_6_)

Figure 33. Supernetwork of G = $\{T_0,T_6,I_2,I_8\}$ networks

```
      <T_6_>
  g2----------g1
   \         / \
    \       /   \
<I_10_> <I_4_> <I_10_>
       \   /      \
        \ /        \
         g1----------g2
             <T_6_>
```

Table 6.10.2 shows an isomorphic mapping G of G onto TIHYP.

[51] The Category 2 G's present another situation altogether. Here, we also find automorphic mappings among networks which express the same G, but these do not correspond directly to any of the forty-eight hyper-operators. Sometimes T_n operators map onto T_n's, and I_n's onto I_n's; and at other times, T_n's map onto I_n's, and vice versa. However, the automorphic mappings of G onto itself do form a group, a permutation group on G's members. As T_0, the identity element, must always map onto itself, these permutations act on the other three members of G. In other words, this permutation group is isomorphic to S_3. Interestingly, like the groups of $\langle T_j\rangle$ and $\langle I_j\rangle$ hyper-operators we have already discussed, S_3 is also dihedral; it is isomorphic to D_6. Therefore, this permutation group's transformations are analogous to, but certainly not the same as, the set of $\{\langle T_0\rangle, \langle T_4\rangle, \langle T_8\rangle, \langle I_j\rangle, \langle I_{j+4}\rangle, \langle I_{j+8}\rangle\}$ hyper-operators.

TABLE 6.10.2 Mapping of F(G) onto TIHYP

Member of G	F	Member of TIHYP
T_0_	<--->	<T_0_>
T_6_	<--->	<T_6_>
I_2_	<--->	<I_10_>
I_8_	<--->	<I_4_>

[6.11] Now we address the relationships among isomorphic G's. As each T/I G is isomorphic to some dihedral group, D_n, the G's which are isomorphic to the same D_n are also isomorphic to each other. By extension, all the various networks which express these isomorphic G's are isographic. We demonstrate that the set TIISO of $\langle T_j \rangle$ and $\langle I_j \rangle$ hyper-operators that map G onto any one of these isomorphic G's is related to TIHYP.

DEFINITION 6.11.1 *TIISO* $= \{\langle c \rangle \mid \langle c \rangle = \langle T_j \rangle$, *or* $\langle c \rangle = \langle I_j \rangle$; *and* $\langle c \rangle G = G'\}$.

If $\langle c \rangle G = G'$, where $\langle c \rangle$ is a member of TIISO, then $\langle c' \rangle G = G'$ for any $\langle c' \rangle$ in TIISO. Of course, G may or may not equal G', and TIHYP may or may not equal TIISO.

[6.12] To discover the relationship between TIHYP and any TIISO, we must first give TTOHYP and TTOISO as the sets of TTO determinants for TIHYP and TIISO, respectively.

DEFINITION 6.12.1 *TTOHYP* $= \{h \mid \langle h \rangle$ *is a member of TIHYP*$\}$.

DEFINITION 6.12.2 *TTOISO* $= \{c \mid \langle c \rangle$ *is a member of TIISO*$\}$.

Then, TTOISO = c(TTOHYP) is a left coset of TTOHYP in TTO_{24}, for any c in TTOISO.

THEOREM 6.12.3 *TTOISO is a left coset c(TTOHYP) of TTOHYP using any c in TTOISO.*

Clearly, if TTOISO = TTOHYP, then c(TTOHYP) = TTOHYP.

[6.13] Therefore, we may give the set TTOLC of all left cosets of TOHYP.

DEFINITION 6.13.1 *TTOLC* $= \{c(TTOHYP) \mid c$ *is a member of* $TTO_{24}\}$.

DEFINITION 6.13.2 $|TTOLC| = 24 / |TTOHYP|$.

The hyper-operators determined by each member of this set map G onto one of its |TTOLC| isomorphic counterparts (in the sense of 6.11.1). Moreover, for each member of TTOLC, the hyper-operators it determines map a network that expresses G onto any of the |HYP| networks (6.5.1) to which it is isographic.

[6.14] Finally, we consider a higher level of structure relating Klein-bottle *Tonnetze*: the hyper-hyper-operators which relate isomorphic sets of hyper-operators.

DEFINITION 6.14.1 $\langle\langle T_x\rangle\rangle\langle T_j\rangle = \langle T_j\rangle$; $\langle\langle T_x\rangle\rangle\langle I_j\rangle = \langle I_{j+x}\rangle$.

DEFINITION 6.14.2 $\langle\langle M_x\rangle\rangle\langle T_j\rangle = \langle T_{5j}\rangle$; $\langle\langle T_x\rangle\rangle\langle I_j\rangle = \langle I_{5j+x}\rangle$.

DEFINITION 6.14.3 $\langle\langle MI_x\rangle\rangle\langle T_j\rangle = \langle T_{7j}\rangle$; $\langle\langle T_x\rangle\rangle\langle I_j\rangle = \langle I_{7j+x}\rangle$.

DEFINITION 6.14.4 $\langle\langle I_x\rangle\rangle\langle T_j\rangle = \langle T_{11j}\rangle$; $\langle\langle T_x\rangle\rangle\langle I_j\rangle = \langle I_{11j+x}\rangle$.

These hyper-hyper-operators may be subjected to an examination similar to the above, and so forth.

[7] CONCLUSIONS

[7.1] So far, we have resisted the impulse to make any significant comparisons between our Klein-bottle *Tonnetze* and the toroidal *Tonnetze* of Cohn et al. Of course, many parallels exist. For example, we might define a group of congruence motions acting on the tessellation of triangles within one of our networks, and this group would share many attributes with that of the LPR-group on the Oettingen/Riemann *Tonnetz*. We also find significant differences. First, a Klein bottle is a one-sided manifold, whereas a torus is two-sided. Therefore, certain cycles on the former visit the same triangle twice: once on the "front" and once on the "back." Second, as we already observed in Note 32, these triangles are not necessarily Cohn functions; they are so only when G is commutative. Moreover, the voice-leading of their corresponding L-, P-, and R-like transformations is not necessarily parsimonious. However, the corresponding transformations are contextual inversions, and, like L, P, and R, they preserve two common tones.

[7.2] In addition to triangular tilings, we might also study the tilings of other shapes on our *Tonnetze*. Of particular salience is the rhombus, which serves as the fundamental region of our G's. The tessellation of rhombi on the Klein bottle's surface is acted on by a group of congruence motions which was a favorite of graphic artist M.C. Escher.[52] Several of his well known works, including "Horseman" (1946), incorporate this group.[53] For our purposes, the four vertices of a rhombus might represent pitch-classes, allowing us to model relationships among tetrachords, using four-node Klumpenhouwer networks.

[7.3] In certain graphic works, Escher metamorphosed one shape into another through a process of gradual distortion. Using the same group as "Horseman," "Sky and Water II" (1938) demonstrates such a process, in which fish gradually become air, and birds become water. Similarly, we may distort pitch-class collections in our *Tonnetze* by incorporating various processes. One example uses an anti-isomorphic permutation of Category 1 *Tonnetze*, and rotates each subsequently higher row in the direction of its T_n arrows. The result is a cycle of weak isographies between shapes in one row and those in the consecutively higher rows.[54]

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[52] Coxeter and Moser (1965, 43) give this group as "p g" (see Note 23). Pólya (1924, 280–81), whose work Escher knew, gives it as "C_2."
[53] See Schattschneider (1990) for an account of the plane symmetry groups used in Escher's periodic drawings.
[54] This rotational scheme is similar to the X transformation of Morris (1998, 188–93), as applied to Perle Spaces. However, Perle Spaces are not conceived of as using glide reflections.

[7.4] Finally, and perhaps most significantly, we address the applicability of this theory to music analysis. We have considered three examples in this study: Schoenberg's "Nacht," Lutosławski's *Funeral Music*, and Crumb's "Gargoyles." Particularly apt are examples that lend themselves well to analysis using Klumpenhouwer networks, which also possess a high degree of common-tone retention among collections, and which use limited set-class material. Such examples are abundant in the body of atonal music, and will suggest further extensions and refinements to the theory of Klein-bottle *Tonnetze*.

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[9] APPENDIX

[9.1] This section contains the proofs of the theorems in the text.

THEOREM 2.6.3 $w = zx$.

Proof. $w = zx$

$= z(z^{-1}w)$	by 2.6.1
$= (zz^{-1})w$	by the associative property
$= w$	Q.E.D.

THEOREM 2.6.4 $z = wx^{-1}$.

Proof. $z = wx^{-1}$

$= w(z^{-1}w)^{-1}$	by 2.6.1
$= w(w^{-1}z)$	by the calculation of inverses
$= (ww^{-1})z$	by the associative property
$= z$	Q.E.D.

COROLLARY 2.6.5 $z = xw$.

Proof. $z = xw$

$= (z^{-1}w)w$	by 2.6.1
$= z^{-1}(ww)$	by the associative property
$= z^{-1}y$	by 2.6.2
$= z^{-1}(zz)$	by 2.6.2
$= (z^{-1}z)z$	by the associative property
$= z$	Q.E.D.

THEOREM 2.9.1 $y^n a = ay^n$, for any a in G, and power of y.

Proof. (A) We begin by showing that y commutes with w.
Specifically, as $y = w^2$ (by 2.6.2), then y is a member of W. Since W is cyclic, it is commutative; hence, y commutes with w. The same argument holds for y and z.

(B) Next, we show that y commutes with x.

$xy = yx$

$= (zz)x$	by 2.6.2
$= z(zx)$	by the associative property
$= zw$	by 2.6.3
$= (xw)w$	by 2.6.5
$= x(ww)$	by the associative property
$= xy$	by 2.6.2

(C) Therefore, we conclude that y commutes with all elements of G, as any member of G may be expressed as a product of powers of w and x. By the same argument, all powers of y commute with every member of G. Q.E.D.

THEOREM 2.9.2 X is normal in G.

Proof. It suffices to show that the conjugate of x by the general element $x^a w^b$ of G is a generator of X; specifically, it is either x or x^{-1}. We find two cases: (1) when b is odd, in which event the conjugate is x^{-1}; and (2) when b is even, wherein it is x.

Case 1: $b = 1$, so b is odd.

$$
\begin{array}{lll}
(x^a w)^{-1}(x)x^a w = x^{-1} & & \\
& = (z^{-1}w)^{-1} & \text{by 2.6.1} \\
& = w^{-1}z & \text{by inversion} \\
& = w^{-1}(xw) & \text{by 2.6.5} \\
& = w^{-1}(x)w & \text{by association} \\
& = w^{-1}(x^{-a}xx^a)w & \text{by def. of cyclic group} \\
& = (w^{-1}x^{-a})x(x^a w) & \text{by association} \\
& = (x^a w)^{-1}x(x^a w) & \text{by inversion} \\
& = (x^a w)^{-1}(x)x^a w & \text{as conjectured}
\end{array}
$$

Case 2: $b = 2$, so b is even.

$$
\begin{array}{lll}
(x^a w^2)^{-1}(x)x^a w^2 = x & & \\
& = z^{-1}w & \text{by 2.6.1} \\
& = (xw)^{-1}w & \text{by 2.6.5} \\
& = w^{-1}x^{-1}w & \text{by inversion} \\
& = w^{-1}(z^{-1}w)^{-1}w & \text{by 2.6.1} \\
& = w^{-1}(w^{-1}z)w & \text{by inversion} \\
& = (w^{-1}w^{-1})zw & \text{by association} \\
& = w^{-2}zw & \text{by composition} \\
& = w^{-2}(xw)w & \text{by 2.6.5} \\
& = w^{-2}x(ww) & \text{by association} \\
& = w^{-2}(x)w^2 & \text{by composition} \\
& = w^{-2}(x^{-a}xx^a)w^2 & \text{by def. of cyclic group} \\
& = (w^{-2}x^{-a})x(x^a w^2) & \text{by association} \\
& = (x^a w^2)^{-1}x(x^a w^2) & \text{by inversion} \\
& = (x^a w^2)^{-1}(x)x^a w^2 & \text{as conjectured}
\end{array}
$$

Then, as b is always the sum of some combination of 1's and 2's, the conjugate of x is either x or x^{-1}, both of which are generators of X. Q.E.D.

THEOREM 2.9.3 If $2 \mid |X|$, then x^k, where $k = |X| / 2$, is a member of the center of G.

Proof. First, x^k commutes with x, as they are members of the same cyclic group. Then, we show

$$
\begin{array}{lll}
wx^k = x^k w & & \\
& = x^{-k}w & \text{by definition of involution} \\
& = z(x^k)z^{-1}w & \text{by 2.9.2 (applied to z)} \\
& = zx^k x & \text{by 2.6.1} \\
& = zx(x^k) & \text{by def. of cyclic group} \\
& = wx^k & \text{by 2.6.3.}
\end{array}
$$

Thus, as x^k commutes with the generators x and w of G, it commutes fully with G. Q.E.D.

THEOREM 2.10.3 $2 \mid |G|$.

Proof. $|XY| = |G| / 2$	by 2.6.2
XY is normal in G	by 2.9.1-2
$|G / XY| = 2$	by definition of quotient groups
Therefore, $2 \mid |G|$.	Q.E.D.

THEOREM 2.11.1 If ab = ba for any a,b in G, then |X| = 2.

Proof. $x = x^{-1}$

$= (z^{-1}w)^{-1}$	by 2.6.1
$= w^{-1}z$	by inversion
$= zw^{-1}$	by the commutative property
$= (xw)w^{-1}$	by 2.6.5
$= x(ww^{-1})$	by the associative property
$= x$	Q.E.D.

THEOREM 3.1.2 $|P| = |W(p)||X(p)| / |W(p) \cap X(p)|$.

Proof. The proof is left for the reader.

THEOREM 3.5.6 For any *Einheit* h, t+(h) and t-(h) have the same GIS-interval content.

Proof. Let (a,b,c) be the interval series of t+(h), regardless of h's parity. Then,

$x^n w^{-1} = w^{-1}x^{-n}$	by extension of 2.6.4-5
$= (x^n w)^{-1}$	by inversion

applied to 3.5.3-4 gives the interval series of the corresponding t-(h) as (a,b^{-1},c^{-1}). Thus, by 3.5.5, both triangles have the same GIS-interval content: $\{e,a,a^{-1},b,b^{-1},c,c^{-1}\}$. Q.E.D.

THEOREM 3.6.1 Triangles of the same *TRI*-class whose nodes are (left) transforms of each other by operations in C_G (2.9.3) have the same interval series.

Proof. Using the variable j from 3.5.1-4,

(1) for any *Einheit* h, h and h′ = y(h) are determined by the same power of x; therefore, we find no variation in the j's of their respective interval-series functions.

(2) If 2 | |X|, then triangles of the same *TRI*-class related by x^k, where k = |X| / 2, have the same interval series. This situation arises because, for any j, 2j = 2(j+k) mod |X|.

(3) We may extend the above conditions to any power of y, and to products of powers of y and x^k. Q.E.D.

THEOREM 4.4.1 Given $w = T_m$ and $x = T_6$ (by 2.11.1). $X \subseteq W$ if there exists some integer i, such that mi = 6.

Proof. $W = \{T_{mi} \mid i \text{ is an integer mod } 12/m\}$. $X = \{T_0,T_6\}$ (by 2.11.1). Therefore, T_0 is a member of both W and X, and their intersection contains at least one element. Furthermore, if there exists an integer i, such that mi = 6, then T_6 is a member of W. Hence, in this case, X of order 2 (by 2.11.1) is a subset of (or equal to) W, and $|W| \geq 2$. Q.E.D.

THEOREM 5.3.3 In any Klein-bottle *Tonnetz* which incorporates T_n and I_n operators, |W| = |Z| = 2.

Proof. (A) If w and z are both I_n operators (Category 1), they both generate involutions (by definition of the T/I group's isomorphism with the dihedral group of order 24, D_{24}. (B) If w and x are both I_n operators (Category 2), then $w^2 = e$ (as above). Now, z^2 must also equal e (by 2.5.1). Therefore, as $w = I_m$, $x = I_n$, and z = xw (by 2.6.4), then $z = I_nI_m = T_{n-m}$. Thus, for this z^2 to equal e, z must be either T_0 or T_6. However, if $z = T_0$, then by 2.6.3, $w = zx = (T_0)x = x$, which we eliminate as w and z are distinct glide reflections. Thus, z must be T_6, of order 2. The same argument holds for x and z as I_n operators; in this case, $w = T_6$. Q.E.D.

THEOREM 5.6.1 Given $w = I_m$, and $x = T_n$ (Category 1), $W(p) \subseteq X(p)$ if there exists some integer i, such that $ni+p = m-p$.

Proof. First, $W(p) = \langle I_m \rangle (p) = \{p,m-p\}$, and $X(p) = \langle T_n \rangle (p) = \{ni+p \mid i \text{ is an integer mod } 12/n\}$. Therefore, p is a member of both W(p) and X(p), and their intersection contains at least one element. Next, if there exists an integer i, such that $ni+p = m-p$, then $m-p$ is also a member of X(p). Hence, in this case, $W(p) \subseteq X(p)$. Q.E.D.

THEOREM 5.7.1 If $z = T_6$ (Category 2), then $|W(p) \cap X(p)| = 1$.

Proof. If $z = T_6$, then w is of the form I_m. This w is always distinct from $x = z^{-1}w$ (by 2.6.1) $= T_6I_m = I_{m+6}$, as no m satisfies $m = m+6$. Then, as $\langle I_m \rangle (p) = \{p,m-p\}$ is not equal to $\langle I_{m+6} \rangle (p) = \{p,(m+6)-p\}$, we observe their trivial intersection, $\{p\}$. Q.E.D.

THEOREM 6.12.3 TTOISO is a left coset c(TTOHYP) of TTOHYP using any c in TTOISO.

Proof. We give $G = \{T_{ni},I_{ni+m} \mid i \text{ is a member of } Z_{12/n}\}$, and $G' = \{T_{nj},I_{nj+m+x} \mid j \text{ is a member of } Z_{12/n}\}$.

Then, $\text{TTOISO} = \{T_{ni'+x},I_{2m+n(i'+j')+x} \mid i' \text{ and } j' \text{ are members of } Z_{12/n}\}$. Furthermore, $\text{TTOHYP} = \{T_{ni},I_{2m+n(i+j)} \mid i,j \text{ are members of } Z_{12/n}\}$ is the TTOISO which carries G onto G. We verify that c(TTOHYP) = TTOISO for any c is a member of TTOISO.

(1) $T_{ni'+x}T_{ni} = T_{(ni'+x)+(ni)} = T_{n(i'+i)+x}$ is a member of TTOISO, for some $(i'+i)$ mod 12/n.

(2) $I_{2m+n(i'+j')+x}T_{ni} = I_{(2m+n(i'+j')+x)-(ni)} = I_{2m+n(i'+j'-i)+x}$ is a member of TTOISO, for some $(i'+j'-i)$ mod 12/n.

(3) $T_{ni'+x}I_{2m+n(i+j)} = I_{(ni'+x)+(2m+n(i+j))} = I_{2m+n(i'+i+j)+x}$ is a member of TTOISO, for some $(i'+i+j)$ mod 12/n.

(4) $I_{2m+n(i'+j')+x}I_{2m+n(i+j)} = T_{(2m+n(i'+j')+x)-(2m+n(i+j))} = T_{n(i'+j'-i-j)+x}$ is a member of TTOISO for some $(i'+j'-i-j)$ mod 12/n.

Thus, TTOISO is a left coset of TTOHYP for any of its members. Q.E.D.

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