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Generalized Contextual Groups

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ABSTRACT: There is a neo-Riemannian-type group, the *generalized contextual group*, that can be constructed for the set of transpositions and inversions of any pitch-class segment, provided the segment contains at least two distinct pitch classes which span an interval other than a tritone. The generalized contextual group is dual to the T/I group for any pitch-class segment satisfying the condition above. This article shows how one can construct a generalized contextual group with symmetrical as well as asymmetrical pitch-class segments (previous work in this area has required asymmetry as a precondition to the construction of neo-Riemannian-type groups). The group is "generalized" in that there is a generalized contextual group for any **Zm**, so **Z12** is just one subtype. We apply a *mod* 12 generalized contextual group (and its dual relation) in a network analysis of Hindemith, *Ludus Tonalis*, Fugue in E. We close with an application of a *mod* 3 generalized contextual group in an analysis of a chorale's chord voicings.

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GENERALIZED CONTEXTUAL GROUPS

THOMAS M. FIORE AND RAMON SATYENDRA

1. Introduction

Though one is accustomed to thinking of the neo-Riemannian L/R group as acting on the set of consonant triads, there is a neo-Riemannian-type group that can be constructed for the set of transpositions and inversions of any pitch-class segment, provided the segment contains at least two distinct pitch classes which span an interval other than a tritone.[1] This paper describes how to construct neo-Riemannian-type groups for any pitch-class segment satisfying this condition.[2] This study is motivated by David Lewin's analysis of Schoenberg, op. 23, no. 3.[3] In that article, Lewin works with a neo-Riemannian-type group that acts on the 24 T- and I-forms of the pitch-class segment $\langle B\flat, D, E, B, C\sharp \rangle$. He proves that a neo-Riemannian-type group which transforms forms of $\langle B\flat, D, E, B, C\sharp \rangle$ is the dual group of the T/I group.[4] In this paper, we show that our neo-Riemannian-type group called the *generalized contextual group* is dual to the T/I group, no matter which initial pitch-class segment is chosen, provided the segment satisfies the condition above. Lewin's neo-Riemannian-type group and the S/W group are special cases of our generalized contextual group.[5] If the unordered collection of pitch classes comprising the pitch-class segment is an asymmetrical pitch-class set, then our theory applies to the set of T- and I-forms of the unordered family of pitch classes as well. This article shows how one can construct neo-Riemannian-type groups with symmetrical as well as asymmetrical pitch-class segments (previous work in this area has required asymmetry as a precondition to the construction of neo-Riemannian-type groups).[6] Our group is called *contextual* because the effect of applying an operation from the group to a pitch-class segment depends on the

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[1]The L/R group is generated by Riemann's leading-tone exchange and relative operations. We follow Morris 1987 in naming an ordered collection of pitch classes a pitch-class segment. Morris allows pitch-class repetitions in pitch-class segments. See Robert Morris, *Composition with Pitch-Classes* (New Haven: Yale University Press, 1987), 64. We also allow repeated pitch classes in a pitch-class segment.

[2]Here "neo-Riemannian-type group" denotes the dual group to the T/I group in a generalized interval system for pitch-class sets. The relevant ideas will be explained in later sections of this paper.

[3]David Lewin, "Transformational Considerations in Schoenberg's Opus 23, Number 3," *preprint*, 2003.

[4]The T/I group, the 24 familiar operations from atonal theory, is generated by T_1 and I.

[5]The S/W group, also called the schritt-wechsel group, is the same as the L/R group. See Henry Klumpenhouwer, "Some Remarks on the Use of Riemann Transformations," *Music Theory Online* 0/9 (1994).

[6]We call a pitch-class set *asymmetrical* if its degree of symmetry is 1. Hook has observed that his triadic results also apply to unordered asymmetrical pitch-class sets; see Julian Hook, *Uniform Triadic Transformations* (Ph.D. diss., Indiana University, 2002), 99.

form of the segment.[7] It is called *generalized* in that, as will be taken up shortly, there is a generalized contextual group for any $\mathbb{Z}_m$, so $\mathbb{Z}_{12}$ is just one subtype. We illustrate our theory with transformational networks drawn from Hindemith, *Ludus Tonalis*, Fugue in E. We also present a *mod* 3 analysis of chord voicings in a chorale passage.

2. Preliminaries

We first summarize the notion of $COMM$-$SIMP$ duality developed in *GMIT*[8] and then describe three theorems.

Lewin has shown that the notion of a *generalized interval system* (GIS) can be restated in terms of a group $SIMP$, where $SIMP$ denotes a group that acts simply transitively[9] on the musical space S of the GIS. A GIS consists of a family S of musical elements, a mathematical group $IVLS$ which consists of the *intervals* of the GIS, and an *interval function* $int : S \times S \to IVLS$ such that:

(1) For all $r, s, t \in S$, we have $int(r, s)int(s, t) = int(r, t)$,
(2) For every $s \in S$ and every $i \in IVLS$, there exists a unique $t \in S$ such that $int(s, t) = i$.

To get from the GIS formulation to the $SIMP$ formulation, one takes $SIMP$ to be the family of transpositions by the intervals of the GIS. There also is a way to construct a GIS with musical space S from a group $SIMP$ which acts simply transitively on S. In what follows we will take advantage of both the GIS and the simply transitive perspectives.

Theorem 2.1. *Suppose we have a GIS with musical space S, interval function $int : S \times S \to IVLS$, and transposition group $SIMP$. Let $COMM$ denote the group consisting of all operations*[10] *on S which commute with all operations in $SIMP$. Then $COMM$ is precisely the family of interval-preserving operations for the GIS.*[11] *The group $COMM$ is called the dual group to $SIMP$ or the dual group to the GIS.*

Theorem 2.2. *Let $COMM$ and $SIMP$ be as in Theorem 2.1. The dual group $COMM$ acts simply transitively on S, and hence determines a GIS whose transpositions are $COMM$. This GIS is called the dual GIS. The dual group to this dual*

[7] Contextual inversion and Q-operations, a kind of contextual transposition, are discussed in David Lewin, *Generalized Musical Intervals and Transformations* [GMIT] (New Haven: Yale University Press, 1987), 147-149, 238-244, and 251-253 as well as in Lewin, *Musical Form and Transformation: 4 Analytic Essays* (New Haven: Yale University Press, 1993), 25-30.

[8] See Lewin 1987, 251-253.

[9] A group action of a group G on a set S is called *simply transitive* if for all $s_1, s_2 \in S$ there is a unique $g \in G$ such that $gs_1 = s_2$. See Lewin 1987, 157-159 for a discussion of the musical connotations of the notion of simple transitivity.

[10] We follow Lewin's terminology. A *transformation on* S is a function $S \to S$ while an *operation on* S is a bijective function $S \to S$. A function is a bijection if and only if it has an inverse.

[11] This statement can be found on page 101 in David Lewin, "Generalized Interval Systems for Babbit's Lists, and for Schoenberg's String Trio," *Music Theory Spectrum* 17/1 (1995): 81-118. Lewin proves one containment on page 50 of *GMIT*, namely that the family of interval-preserving operations is contained in $COMM$. In particular, Theorem 3.4.10 of *GMIT* states that any transposition operation commutes with any interval-preserving operation. He proves it by fixing a *ref.* We present a self-contained proof of Theorem 2.1 in the appendix which does not rely on choosing a *ref.*

GIS is $SIMP$, i.e. the elements of $SIMP$ are the interval-preserving operations for the dual GIS.

Theorem 2.3. *Let $COMM$, $SIMP$, and $IVLS$ be as in Theorem 2.1. Then $SIMP$ is anti-isomorphic to $IVLS$, while $COMM$ is isomorphic to $IVLS$.*[12]

From Theorem 2.3 we make an observation we will need to use later: all three groups–$COMM$, $SIMP$, and $IVLS$–have the same number of elements. Also from Theorem 2.3 we note that composing with $z \mapsto z^{-1}$ in an appropriate manner shows that the groups $COMM$, $SIMP$, and $IVLS$ are all isomorphic to one another.[13] The following two examples illustrate the foregoing terms and concepts.

Example 1. This example considers the L/R group generated by Riemann's leading-tone-exchange and relative operations. Let S be the family of 24 consonant triads. The T/I group acts simply transitively on S and hence determines a GIS. It can be shown that the L/R group consists precisely of those operations that commute with the T/I operations.[14] We can therefore say the L/R group is the dual group to the T/I group. Theorem 2.1 tells us that the operations in the L/R group are precisely the interval-preserving operations of the GIS determined by the T/I group. In this context, we say that the T/I group is $SIMP$ and the L/R group is $COMM$. Since $COMM$ and $SIMP$ are isomorphic, as previously noted, we conclude that T/I and L/R are isomorphic. We know that T/I has 24 elements: 12 transpositions and 12 inversions.[15] From the isomorphism we conclude that L/R has 24 elements also.[16]

The operations L and R are *contextual* in that the action of each depends upon the form of the object being acted upon. For instance, the result of the L operation is a major or minor triad depending upon whether the triad acted upon is minor or major. We therefore say in the case of the L/R group that the mode of the triad is the context which determines the action of the operations L and R.

Example 2. Lewin's analysis of Schoenberg, op. 23, no. 3, is the source of this example, which is a specific instance of a generalized contextual group acting on a space other than the 24 consonant triads.[17] Let S be the 24 forms of the pitch-class segment $\langle B\flat, D, E, B, C\sharp \rangle$. The T/I group acts simply transitively on S and hence comprises a $SIMP$ and determines a GIS. Lewin proceeds by defining contextual

[12]This follows from Lewin 1987, 46-48. Lewin's Theorem 3.4.2 states that $SIMP$ is anti-isomorphic to $IVLS$. Theorems 3.4.7 and 3.4.5 together imply that the group of interval-preserving operations is isomorphic to $IVLS$. Finally, Theorem 2.1 of this paper says that $COMM$ is precisely the group of interval-preserving operations, hence $COMM$ is isomorphic to $IVLS$.

[13]Anti-isomorphic groups are isomorphic by composing an anti-isomorphism with $z \mapsto z^{-1}$ to get an isomorphism.

[14]This algebraic statement about commutativity is proved in Hook 2002, 93-96.

[15]"Transposition" has two meanings in this example: the usual pitch-class transposition on one hand and GIS transposition on the other. GIS transpositions include all elements of the T/I group, hence the usual pitch-class inversions and pitch-class transpositions may all be regarded as GIS transpositions.

[16]The 24 major and minor chords and P, L, R operations fit together to form a graph on a torus as documented in Jack Douthett and Peter Steinbach, "Parsimonious Graphs: A Study in Parsimony, Contextual Transformations, and Modes of Limited Transposition," *Journal of Music Theory* 42/2 (1998): 241-263. It would be interesting to see how our generalized contextual group relates to the Klein bottle counterparts of P, L, R mentioned in Robert W. Peck, "Klein Bottle *Tonnetze*," *Music Theory Online* 9/3 (2003), 37.

[17]Lewin 2003.

K-, J-, and L-inversion as well as operations Q_i. He then shows that Q_1 and any one of the contextual inversions J, K, or L generate the dual group $COMM$. He proves that the group generated by Q_1 and a contextual inversion is $COMM$ by testing its elements for commutativity with elements from the T/I group. We will be using this procedure of testing for commutativity later.

The $COMM - SIMP$ duality and these two examples have motivated us to use pitch-class segments in our construction of the contextual group.

3. The Definition of the Generalized Contextual Group

We now generalize Example 2 to pitch-class segments in the $\mathbb{Z}_{12}$ universe of any length. For the sake of concreteness we work in $\mathbb{Z}_{12}$, although later we will show how to work in $\mathbb{Z}_m$ for any m. The reader may replace all occurrences of 12 by m and 24 by $2m$ in this section for the more general statements.

Let $X = \langle x_1, ..., x_n \rangle$ be a pitch-class segment with entries in $\mathbb{Z}_{12}$. We make the following assumption on X (the *tritone condition*):

- there are two distinct pitch classes x_q, x_r in X which span an interval other than a tritone.

For example, permissible X are $\langle C, F\sharp, D, D \rangle$ and $\langle E, E, F \rangle$. The pitch-class segment $\langle C\sharp, G \rangle$ is *not* allowed. We make the usual identifications ($C = 0, C\sharp = 1$, and so forth) to work in the $\mathbb{Z}_{12}$ universe. Throughout this section let X be a fixed pitch-class segment satisfying the condition above. Let S denote the family of 24 pitch-class segments that are obtained by transposing and inverting the pitch-class segment X. For example, the pitch-class segment $\langle T_1x_1, T_1x_2, \ldots, T_1x_n \rangle$ and the pitch-class segment $\langle I_3x_1, I_3x_2, \ldots, I_3x_n \rangle$ are elements of S. There are 24 elements in S because of the assumption on X made above. The T/I group acts simply transitively on S and hence defines a GIS.

Now we define operations K and Q_i on S as follows where $i \in \mathbb{Z}_{12}$. For a pitch-class segment $Y = \langle y_1, \ldots, y_n \rangle \in S$, let

$$\begin{aligned} K(Y) &:= I_{y_1+y_2}(Y) \\ &= \langle I_{y_1+y_2}(y_1), I_{y_1+y_2}(y_2), I_{y_1+y_2}(y_3), \ldots, I_{y_1+y_2}(y_n) \rangle \\ &= \langle y_2, y_1, I_{y_1+y_2}(y_3), \ldots, I_{y_1+y_2}(y_n) \rangle. \end{aligned}$$

Recall that $I_{y_1+y_2}$ is the unique inversion which exchanges y_1 and y_2. Thus $K(Y)$ is that inverted form of Y that has y_1 and y_2 interchanged.

For instance, let $X = Y = \langle 2, 11, 4, 9 \rangle = \langle D, B, E, A \rangle$. Then

$$\begin{aligned} K(Y) &= I_{2+11}(Y) \\ &= \langle I_1(D), I_1(B), I_1(E), I_1(A) \rangle \\ &= \langle B, D, A, E \rangle, \end{aligned}$$

because the inversion which swaps D and B is $I_1 = I_{2+11}$. Notice that $\langle B, D, A, E \rangle$ is that inverted form of $\langle D, B, E, A \rangle$ which has the first two entries swapped.

We now return to the general discussion for an abstract pitch-class segment $X = \langle x_1, ..., x_n \rangle$ as above. Since $I_{y_1+y_2}$ has order 2, we see that K is its own inverse and hence K is actually an operation. It may seem that our construction of the contextual group depends on choosing y_1, y_2 rather than y_3, y_4 etc., but we will soon see that the generalized contextual group is independent of this choice. Note that the pitch-class segment was essential to the definition of K. We cannot define

K so easily if we did not have an ordering of the pitch-class set underlying X.[18] This is one advantage of working with forms of a pitch-class segment rather than forms of an unordered pitch-class set.

Let Q_i denote the transformation defined by

$$Q_i(Y) := \begin{cases} \langle T_i(y_1), T_i(y_2), \ldots T_i(y_n)\rangle \text{ if } Y \text{ is a } T\text{-form of } X \\ \langle T_{-i}(y_1), T_{-i}(y_2) \ldots, T_{-i}(y_n)\rangle \text{ if } Y \text{ is an } I\text{-form of } X. \end{cases}$$

In other words, Q_i is the transformation which transposes T-forms of X "up" by the interval i and transposes I-forms of X "down" by the interval i.

For instance, let $X = Y = \langle 2, 11, 4, 9\rangle = \langle D, B, E, A\rangle$. Then $Q_{10}(Y) = \langle C, A, D, G\rangle$. On the other hand,

$$\begin{aligned} Q_{10}(I_{11}(Y)) &= Q_{10}\langle A, C, G, D\rangle \\ &= \langle B, D, A, E\rangle. \end{aligned}$$

In other words, Q_{10} sends Y, a T-form of X, "up" 10 semitones, whereas Q_{10} sends $I_{11}(Y)$, an I-form of X, "down" by 10 semitones. We will return to this in the music analysis in Section 6.

Returning now to to the general discussion for an abstract pitch-class segment $X = \langle x_1, ..., x_n\rangle$, we see that Q_i has inverse Q_{-i} and hence Q_i is an operation. We also notice that Q_i is the composition of Q_1 with itself i times if $i = 0, 1, \ldots, 11$. This concludes the definitions of operations K and Q_i on S.[19]

To work with the twenty-four forms of an asymmetrical pitch-class set X', choose an ordering X of the pitch-class set and define K and Q_i as above. Then K and Q_i induce operations K' and Q'_i on the set of forms of the unordered pitch-class set X' by ignoring the ordering.

Definition 3.1. We call the group of operations on S generated by Q_1 and K the *generalized contextual group*. We sometimes write *contextual group* for short.

The L/R group is a special case of the generalized contextual group: if one selects pitch-class segment $\langle 0, 4, 7\rangle$ for X then the generalized contextual group turns out to be the L/R group and K-inversion turns out to be the R (relative) operation of the L/R group.[20] We will see later that the generalized contextual group is dual to the T/I group acting on S, just as the neo-Riemannian group is dual to the T/I group acting on consonant triads. It is in this sense that we refer to the contextual

[18]Kochavi overcomes this difficulty by choosing canonical representatives and describing contextual inversions in terms of indexing functions. See Jonathan Kochavi, "Some Structural Features of Contextually-Defined Inversion Operators," *Journal of Music Theory* 42/2 (1998): 307-320.

[19]One might call Q_i a schritt and Q_iK a wechsel. Indeed, whenever $X = \langle 0, 4, 7\rangle$ these are the traditional schritts and wechsels. Hook 2002 gives a similar description for the case of set-class [037] in his work on uniform triadic transformations. For more on schritts and wechsels in transformational systems, see Klumpenhouwer 1994, and Brian Hyer, "Reimag(in)ing Riemann," *Journal of Music Theory* 39/1 (1995): 101-138.

[20]Because the generalized contextual group is defined in terms of pitch-class segments and not pitch-class sets, in comparing the L/R group to the generalized contextual group, we interpret the L/R group action on pitch-class segments as follows. We first look at what an L/R operation does to the unordered pitch-class set underlying the pitch-class segment and then choose the appropriate ordering of the output pitch-class set to make it an element of S. For example, for a major triad, $R\langle 0, 4, 7\rangle = \langle 4, 0, 9\rangle$ and *not* $\langle 0, 9, 4\rangle$ because $\langle 4, 0, 9\rangle \in S$ while $\langle 0, 9, 4\rangle \notin S$. Similarly, for a minor triad, $R\langle 0, 8, 5\rangle = \langle 8, 0, 3\rangle$ and *not* $\langle 0, 3, 8\rangle$ because $\langle 8, 0, 3\rangle \in S$ while $\langle 0, 3, 8\rangle \notin S$. It is in this sense that we mean K-inversion turns out to be the R operation.

group as a neo-Riemannian-type group. The contextual group is analogous to the neo-Riemannian group also in that K is a contextual inversion just as are L, R, and P.

Once we know that the contextual group is dual to the T/I group, we can conclude that they are isomorphic and that the generalized contextual group has 24 elements. At this point however, since we will not prove the dual relation until Section 4, we give only the following lemma. Although the lemma is formulated for the $\mathbb{Z}_{12}$ universe, it can be restated in the $\mathbb{Z}_m$ universe by replacing 24 by $2m$.

Lemma 3.1. *The generalized contextual group has at least 24 elements.*

Proof: To show two operations f, g (with the same domain and range) are distinct, we must find a sample x such that $f(x) \neq g(x)$. Following this method, we prove that there are at least 24 distinct elements by evaluating the operations $Q_0, Q_1, \ldots, Q_{11}, Q_0K, Q_1K, \ldots, Q_{11}K$ in the generalized contextual group on the initial pitch-class segment X from above and noting that the outputs are unequal.

The pitch-class segments $Q_0(X), Q_1(X), \ldots, Q_{11}(X)$ are the 12 T-forms of X. The pitch-class segments $Q_0(K(X)), Q_1(K(X)), \ldots, Q_{11}(K(X))$ are the 12 T-forms of $K(X)$, and are therefore the 12 I-forms of X. The 12 T-forms of X are all distinct from the 12 I-forms of X as a consequence of our condition on X, so $Q_0(X), Q_1(X), \ldots, Q_{11}(X)$, $Q_0(K(X)), Q_1(K(X)), \ldots, Q_{11}(K(X))$ are 24 distinct pitch-class segments. We conclude that $Q_0, Q_1, \ldots, Q_{11}$, $Q_0K, Q_1K, \ldots, Q_{11}K$ are 24 distinct elements of the generalized contextual group. Hence the generalized contextual group has at least 24 elements.

□

In this section we have described the generalized contextual group in terms of operations Q_i and K, which are similar to schritts and wechsels. In the next section we will find other descriptions of the generalized contextual group in terms of the GIS theory outlined in Section 2.

4. The Generalized Contextual Group is Dual to the T/I Group

In the notation of the previous section, we have the T/I group acting simply transitively on S, the family of all T- and I-forms of the pitch-class segment X. Therefore this group action determines a GIS where $SIMP$ is the T/I group. As usual, $COMM$ denotes the family of all operations on S that commute with all operations in $SIMP$, i.e. that commute with all T/I operations. We show that $COMM$ is the generalized contextual group in the following two proofs, and then draw some conclusions in the form of corollaries. We conclude that the generalized contextual group is isomorphic to the dihedral group of order 24. Note that the T/I group and the generalized contextual group are subgroups of the symmetric group on the set S. Since the T/I group and the generalized contextual group are isomorphic to the dihedral group of order 24, these two subgroups give rise to two group actions of the dihedral group of order 24 on the set S.[21]

[21]The abstract group structure of the *mod* 12 generalized contextual group is no different than the abstract group structure of the S/W group or the L/R group. They are all isomorphic to the dihedral group of order 24. However, the elements of the generalized contextual group have a different domain and range than the elements of the L/R group, hence the groups are not equal. The *mod* 12 generalized contextual group gives rise to a group action of the dihedral group of order 24 on the the set of forms of a pitch-class segment, while the L/R group gives rise to a group

Lemma 4.1. *Every element of the generalized contextual group commutes with every element of $SIMP$, i.e. the generalized contextual group is contained in $COMM$.*

Proof: It suffices to check that the generators Q_1 and K of the contextual group commute with the generators T_1 and I_0 of $SIMP$. Since Q_1 performs T_1 on T-forms but performs T_{11} on I-forms–in other words, since it has two different effects depending on the input–we consider Q_1's action on T- and I-forms as separate cases for the verifications involving Q_1 below.

- We show Q_1 and T_1 commute:
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is a T-form of X. Then $Q_1T_1(Y) = T_1T_1(Y) = T_1Q_1(Y)$.
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is an I-form of X. Then $Q_1T_1(Y) = T_{11}T_1(Y) = T_1T_{11}(Y) = T_1Q_1(Y)$.
  Hence Q_1 and T_1 commute.
- We show Q_1 and I_0 commute:
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is a T-form of X. Then $Q_1I_0(Y) = T_{11}I_0(Y) = I_0T_1(Y) = I_0Q_1(Y)$.
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is an I-form of X. Then $Q_1I_0(Y) = T_1I_0(Y) = I_0T_{11}(Y) = I_0Q_1(Y)$. Hence Q_1 and I_0 commute.
  The following example illustrates the commutativity of Q_1 and I_0. If $X = Y = \langle D, B, E, A \rangle$, then

$$\begin{aligned} Q_1I_0(Y) &= Q_1\langle B\flat, D\flat, A\flat, E\flat \rangle \\ &= \langle A, C, G, D \rangle \\ &= I_0\langle E\flat, C, F, B\flat \rangle \\ &= I_0Q_1(Y). \end{aligned}$$

  Now we leave this example and return to the general proof.
- We show K and T_1 commute:
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is any form of X. Then

$$\begin{aligned} KT_1(Y) &= I_{y_1+1+y_2+1}\langle y_1 + 1, \ldots, y_n + 1 \rangle \\ &= \langle -(y_1 + 1) + y_1 + y_2 + 2, \ldots, -(y_n + 1) + y_1 + y_2 + 2 \rangle \\ &= \langle -y_1 + y_1 + y_2 + 1, \ldots, -y_n + y_1 + y_2 + 1 \rangle \\ &= T_1\langle -y_1 + y_1 + y_2, \ldots, -y_n + y_1 + y_2 \rangle \\ &= T_1I_{y_1+y_2}\langle y_1, \ldots, y_n \rangle \\ &= T_1K\langle y_1, \ldots, y_n \rangle \\ &= T_1K(Y). \end{aligned}$$

  Hence K and T_1 commute.

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action of the dihedral group on the set of major and minor triads. In Section 6 we will depart from the traditional L/R group in yet another way and obtain group actions of other dihedral groups on pitch-class segments with constituents outside of the *mod* 12 universe.

- We show K and I_0 commute:
  Suppose $Y = \langle y_1, \ldots, y_n \rangle$ is any form of X. Then

$$\begin{aligned} KI_0(Y) &= K\langle -y_1, \ldots, -y_n \rangle \\ &= I_{-y_1-y_2}\langle -y_1, \ldots, -y_n \rangle \\ &= \langle y_1 - y_1 - y_2, \ldots, y_n - y_1 - y_2 \rangle \\ &= I_0\langle -y_1 + y_1 + y_2, \ldots, -y_n + y_1 + y_2 \rangle \\ &= I_0 I_{y_1+y_2}\langle y_1, \ldots, y_n \rangle \\ &= I_0 K\langle y_1, \ldots, y_n \rangle \\ &= I_0 K(Y). \end{aligned}$$

  Hence K and I_0 commute.

We conclude that the generators of the respective groups commute and that the generalized contextual group is contained in the group $COMM$. □

Theorem 4.2. *The commuting group $COMM$ is the generalized contextual group, i.e. the generalized contextual group is the dual group for the T/I group.*

Proof: By the remarks in the Section 2, $COMM$ and $SIMP$ have the same number of elements because they are isomorphic. Since $SIMP$ is the T/I group, it has 24 elements. Hence $COMM$ has 24 elements. By Lemma 4.1 the generalized contextual group is contained in $COMM$. But by Lemma 3.1 the generalized contextual group has at least 24 elements. Thus the generalized contextual group is equal to $COMM$. □

Corollary 4.3. *The generalized contextual group is isomorphic to the T/I group and hence is dihedral of order 24.*

Proof: By the general remarks in the preliminaries, the groups $COMM$ and $SIMP$ are isomorphic. Here $COMM$ is the contextual group and $SIMP$ is the T/I group. This proves the corollary. □

Corollary 4.4. *The definition of the generalized contextual group does not depend on the choice of y_1 and y_2.*

Proof: Let K' be the operation defined like K, except that K' swaps y_k and y_ℓ instead of swapping y_1 and y_2. If we write y_k and y_ℓ instead of y_1 and y_2 in the proofs above then we see that the group generated by Q_1 and K' is the commuting group for the T/I group. But the generalized contextual group is also the commuting group. Therefore the group generated by Q_1 and K' is actually *equal* to (not just isomorphic to) the generalized contextual group.[22] Hence, the choice y_1 and y_2 was irrelevant, any y_k and y_ℓ would do.[23] Note that although K and K' are different, the group generated by Q_1 and K is actually *equal* to the group generated by Q_1 and K', namely they are both the generalized contextual group. □

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[22] See the previous footnote on the distinction between equality and isomorphism.

[23] The notes y_k and y_ℓ could even be the same, i.e. $k = \ell$.

Corollary 4.5. *The generalized contextual group is the group of interval-preserving operations for the generalized interval system associated to the simply transitive T/I group action on the set S.*

Proof: By Theorem 2.1 the group of interval-preserving operations is the same as the commuting group $COMM$. □

Corollary 4.6. *The generalized contextual group acts simply transitively on S.*

Proof: The generalized contextual group is the dual group $COMM$ to $SIMP = T/I$. The corollary then follows from Theorem 2.2. □

To summarize, given a pitch-class segment X satisfying the tritone condition, we have described the generalized contextual group of operations on the set S of forms of X in three equivalent ways. It is the group generated by the "schritt" Q_1 and the "wechsel" K. On the other hand the generalized contextual group consists precisely of those operations on S which commute with all elements of the T/I group on S. More abstractly, it is the simply transitive group belonging to the dual GIS, i.e. it is the group of interval-preserving operations for the GIS associated to the simply transitive group action of the T/I group on S.

5. Relationship to the Literature

Special cases of the generalized contextual group can be found in the literature. In this section we consider some recent examples in neo-Riemmanian theory. We also adapt Kochavi's results to pitch-class segments and give corollaries about the algebraic structure of the group generated by Q_iK and the transposition T_1.

David Lewin's analysis of Stockhausen's Klavierstück III considers contextual inversions outside of the realm of neo-Riemannian transformations.[24] It applies J-inversion to forms of the segment $X = \langle x_1, x_2, x_3, x_4, x_5\rangle = \langle 0, 1, 2, 3, 6\rangle$. The pitch-class segment X satisfies our condition because 0 and 1 in X are distinct pitches which span an interval other than a tritone. Lewin introduces the notation

$$P = \langle 8, 9, 10, 11, 2\rangle$$
$$p = I_7(P) = \langle 11, 10, 9, 8, 5\rangle$$

for his analysis. Let S be the collection of all forms of X. Lewin defines the J operation on S as follows: for $Y \in S$, the pitch-class segment $J(Y)$ is that unique inverted form of Y which preserves the chromatic tetrachord subset of Y. In our notation,

$$J(Y) = I_{y_1+y_4}Y.$$

Thus

$$\begin{aligned} J(P) &= I_{8+11}\langle 8, 9, 10, 11, 2\rangle \\ &= I_7\langle 8, 9, 10, 11, 2\rangle \\ &= \langle 11, 10, 9, 8, 5\rangle \\ &= p \end{aligned}$$

as Lewin requires. It follows from our formalism that J and Q_1 generate the generalized contextual group, i.e., the commuting group of the T/I group. Lewin goes on to conclude that the twenty-four operations T_m and $J_n := T_nJ$ commute

[24] Lewin 1993, 26.

with each other, which our formalism already implies. One should note, however, that the 24 operations of the form T_m and J_n do *not* form the generalized contextual group, although they do form a simply transitive group. This group generated by T_1 and J is abelian, and hence is not isomorphic to the generalized contextual group.[25] Lewin goes on in the analysis to explore the role of the contextual inversion J in the piece.

Contextual inversions in the set class [0258] were studied by Adrian Childs and Edward Gollin.[26] Childs makes a strong case for incorporating dominant sevenths and half-diminished sevenths within neo-Riemannian theory by considering examples from Wagner and Chopin. He notes that a triadic analysis overlooks voice leading and other subtleties. Childs and Gollin both develop the generalized contextual group for the forms of the pitch-class segment $X = \langle 0, 4, 7, 10 \rangle$, i.e. for the set class of dominant-seventh chords and half-diminished seventh chords.[27] However, they do not highlight a GIS perspective when drawing an analogy to the neo-Riemannian group for major and minor triads. Gollin does mention the generalized contextual group for triads and for set class [0258]: "The exhaustive composition of the contextual inversions of any asymmetrical tetrachordal (or trichordal) set class will similarly yield an S/W group isomorphic to the S/W groups acting on triads or [0258] tetrachords."[28]

Childs' operations are contextually defined, that is, are elements of the generalized contextual group. In his notation, the operation $S_{m(n)}$ applied to a dominant seventh results in holding the two pitches of interval class m constant and moving the two pitches of interval class n in the same direction. For example,

$$S_{2(3)}(\text{F dominant seventh}) = \text{F half-diminished seventh}.$$

We give a dictionary below relating the $S_{m(n)}$ notation to the generalized contextual group notation by evaluating the $S_{m(n)}$ operations on a form $Y = \langle y_1, y_2, y_3, y_4 \rangle$ of $X = \langle 0, 4, 7, 10 \rangle$ below.

$$\begin{aligned} S_{2(3)}(Y) &= I_{y_1+y_4}(Y) \\ S_{3(2)}(Y) &= I_{y_2+y_3}(Y) \\ S_{3(4)}(Y) &= I_{y_3+y_4}(Y) \\ S_{4(3)}(Y) &= I_{y_1+y_2}(Y) \\ S_{5(6)}(Y) &= I_{y_1+y_3}(Y) \\ S_{6(5)}(Y) &= Q_6 I_{y_2+y_4}(Y) \end{aligned}$$

Thus, Childs' operations $S_{m(n)}$ are elements of the generalized contextual group for the pcseg $X = \langle 0, 4, 7, 10 \rangle$.

Julian Hook's exhaustive study of *uniform triadic transformations* (UTTs) considers many aspects of neo-Riemannian theory.[29] Among other things, he classifies

[25]See John Clough, "A Rudimentary Geometric Model for Contextual Transposition and Inversion," *Journal of Music Theory* 42/2 (1998): 297-306, for more on such recombinations between the T/I group and S/W group.

[26]See Adrian P. Childs, "Moving Beyond Neo-Riemannian Triads: Exploring a Transformational Model for Seventh Chords," *Journal of Music Theory* 42/2 (1998): 191-193, and Edward Gollin, "Some Aspects of Three-Dimensional *Tonnetze*," *Journal of Music Theory* 42/2 (1998): 195-206.

[27]Although Childs and Gollin do not work with pitch-class segments, they are implicit in their analysis.

[28]Gollin 1998, 204.

[29]Hook 2002.

all the simply transitive groups of UTTs, elegantly proves many results of neo-Riemannian theory, and sets up a standardized nomenclature. Hook notes that his work extends to asymmetrical pitch-class sets and tone rows, although he does not consider arbitrary pitch-class segments. Contextual inversions do not play an essential role in his work. In his consolidation of previous research, Hook reinterprets Childs' and Gollin's work as well as Lewin's analysis of Stockhausen.[30]

Jonathan Kochavi investigates *contextually-defined inversion operations* (cio's) on the set of forms of an asymmetrical pitch-class set.[31] He selects canonical representatives from each form in a uniform way and introduces indexing functions in order to relate cio's to traditional inversion operations. In his examination of groups generated by a cio I and T_1, Kochavi elegantly relates the commutativity of such groups to the structure of the indexing function.[32] In fact, such groups are either $\mathbb{Z}_{12} \bigoplus \mathbb{Z}_2$ or $\mathbb{Z}_{24}$ in the commutative case. The structure of the indexing function tells us which group it is.

In these concluding paragraphs of our literature review, we reformulate Kochavi's work in terms of pitch-class segments. The introduction of pitch-class segments rather than pitch-class sets makes the asymmetry assumption superfluous. As before, let $X = \langle x_1, ..., x_n \rangle$ be a pitch-class segment such that

- there are two distinct pitch classes x_q, x_r in X which span an interval other than a tritone.

Again, we let S denote the set of T- and I-forms of X.

Definition 5.1. An operation $I : S \to S$ is a *contextually-defined inversion operation* (cio) if and only if $I(Y)$ and Y are inversionally related for all $Y \in S$.

Examples of cio's include $I_0, \ldots, I_{11}, Q_0K, \ldots, Q_{11}K$. Note that Q_i is *not* a cio. As a result there are many cio's that are not in the generalized contextual group and only half of the elements of the generalized contextual group are cio's.

Definition 5.2. Choose $1 \leq s \leq n$. Then x_s is called the *canonical representative of* X. The *canonical representative of* $T_i(X)$ is $x_s + i$ and the *canonical representative of* $I_i(X)$ is $i - x_s$. In particular, the *canonical representative* y *of* $Y = \langle y_1, \ldots, y_n \rangle \in S$ is y_s.

Definition 5.3. Let $I : S \to S$ be a cio. Then its *indexing function* is the unique map $f : S \to \mathbb{Z}_{12}$ which satisfies

$$I(Y) = I_{f(Y)}(Y).$$

The two *indexing function pieces* f_1 and f_2 of I are the maps $f_1, f_2 : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$f(Y) = f_1(y) \text{ if } Y \text{ is a transposed form of } X$$
$$f(Y) = f_2(y) \text{ if } Y \text{ is an inverted form of } X$$

where y is the canonical representative of Y.

Although the indexing function pieces depend on the choice of $1 \leq s \leq n$ in the definition of canonical representative, the indexing function itself is independent of this choice. We use this fact in equations (1) and (2) to prove Lemma 5.1. This

[30] Hook 2002, 98-104.

[31] Kochavi 1998.

[32] In this part of the paper we follow Kochavi's convention that the letter I without an index refers to a cio and does *not* refer to traditional inversion (Kochavi 1998, 308).

Lemma tells us how to navigate between indexing function pieces with respect to two choices s and s'. As we will see, the indexing function pieces change in a predictable way if we choose s' instead of s.

Lemma 5.1. *Let $1 \leq s, s' \leq n$ and let $I : S \to S$ be a cio. Let f_1 and f_2 denote the indexing function pieces for I with respect to the choice of s in the definition of canonical representative. Let f_1' and f_2' denote the indexing function pieces for I with respect to the choice of s' in the definition of canonical representative. Let $j := x_s - x_{s'}$. Then*

$$f_1'(z) = f_1(z + j)$$
$$f_2'(z) = f_2(z - j)$$

for all $z \in \mathbb{Z}_{12}$.

Proof: Suppose Y is a transposed form of X. Then $j = x_s - x_{s'} = y_s - y_{s'}$, so

$$\begin{aligned} f_1'(y_{s'}) &= f(Y) \\ &= f_1(y_s) \\ &= f_1(y_{s'} + j). \end{aligned} \tag{1}$$

Since every $z \in \mathbb{Z}_{12}$ is the canonical representative of some transposed form of X, we have $f_1'(z) = f_1(z + j)$ for all $z \in \mathbb{Z}_{12}$.

Suppose Y is an inverted form of X. Then $j = x_s - x_{s'} = y_{s'} - y_s$, so

$$\begin{aligned} f_2'(y_{s'}) &= f(Y) \\ &= f_2(y_s) \\ &= f_2(y_{s'} - j). \end{aligned} \tag{2}$$

Since every $z \in \mathbb{Z}_{12}$ is the canonical representative of some inverted form of X, we have $f_2'(z) = f_2(z - j)$ for all $z \in \mathbb{Z}_{12}$. □

Kochavi's first result adapted to pitch-class segments can be formulated as the following Theorem, in which we have added a second statement about indexing function pieces.

Theorem 5.2. *Let $I : S \to S$ be a cio with indexing function pieces f_1 and f_2 with respect to the choice of s in the definition of canonical representative.*

(1) *The group G of operations on S generated by the cio I and the transposition operation T_1 is commutative if and only if f_1 and f_2 can be expressed as*

$$f_1(z) = 2z + c_1$$
$$f_2(z) = 2z + c_2$$

for some $c_1, c_2 \in \mathbb{Z}_{12}$.

(2) *Let f_1' and f_2' denote the indexing function pieces for I with respect to the choice of s' in the definition of canonical representative. If f_1 and f_2 can be expressed as above, then f_1' and f_2' can be expressed as*

$$f_1'(z) = 2z + c_1'$$
$$f_2'(z) = 2z + c_2'$$

for some $c_1', c_2' \in \mathbb{Z}_{12}$. Moreover $c_1' + c_2' = c_1 + c_2$.

Proof: Kochavi proves the first statement.[33] The second statement follows from the first, but here we instead prove it directly. Suppose f_1 and f_2 have the indicated form. Let $c'_1 := 2j + c_1$ and $c'_2 := -2j + c_2$. By Lemma 5.1, we have

$$\begin{aligned} f'_1(z) &= f_1(z+j) \\ &= 2(z+j) + c_1 \\ &= 2z + 2j + c_1 \\ &= 2z + c'_1 \end{aligned}$$

and we also have

$$\begin{aligned} f'_2(z) &= f_2(z-j) \\ &= 2(z-j) + c_2 \\ &= 2z - 2j + c_2 \\ &= 2z + c'_2. \end{aligned}$$

Therefore f'_1 and f'_2 also have the indicated form.

Note that if c_1, c_2, c'_1, and c'_2 exist that give f_1, f_2, f'_1, and f'_2 the form indicated in the Theorem, then they are unique. Therefore they must be related by

$$\begin{aligned} c'_1 &= 2j + c_1 \\ c'_2 &= -2j + c_2, \end{aligned}$$

from which we see $c'_1 + c'_2 = c_1 + c_2$. □

Corollary 5.3. *The group of operations on S generated by the cio Q_iK and the transposition operation T_1 is commutative.*[34]

Proof: Let the canonical representative of Y be $y := y_1$. The second statement of Theorem 5.2 allows us to make this choice. Let $c_1 := x_2 - x_1 - i$ and $c_2 := x_1 - x_2 + i$. Suppose that Y is a transposed form of X. Then $y_2 - y_1 = x_2 - x_1$ and

$$\begin{aligned} Q_iK(Y) &= T_{-i}I_{y_1+y_2}(Y) \\ &= I_{y_1+y_2-i}(Y) \\ &= I_{2y_1+y_2-y_1-i}(Y) \\ &= I_{2y+x_2-x_1-i}(Y) \\ &= I_{2y+c_1}(Y). \end{aligned}$$

Similarly, suppose Y is an inverted form of X. Then $y_2 - y_1 = x_1 - x_2$ and

$$\begin{aligned} Q_iK(Y) &= T_iI_{y_1+y_2}(Y) \\ &= I_{y_1+y_2+i}(Y) \\ &= I_{2y_1+y_2-y_1+i}(Y) \\ &= I_{2y+x_1-x_2+i}(Y) \\ &= I_{2y+c_2}(Y). \end{aligned}$$

Hence $f_1(z) = 2z + c_1$ and $f_2(z) = 2z + c_2$ for the $c_1, c_2 \in \mathbb{Z}_{12}$ defined above and the Corollary follows from Theorem 5.2. □

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[33] See Theorem 1 in Kochavi 1998, 310.

[34] This corollary follows immediately from the fact that Q_iK belongs to the commuting group of the T/I group. However, it is nice to see that it also follows from Kochavi 1998.

The second statement of Theorem 5.2 tells us that the specific choice of canonical representative of Y in Lemma 5.4 and Theorem 5.6 is immaterial because $c_1 + c_2$ is independent of the choice of s. Therefore we do not mention the choice of canonical representative in the hypotheses of Lemma 5.4 and Theorem 5.6.

Lemma 5.4. *(Kochavi) Let $I : S \to S$ be a cio. Suppose that the group G in Theorem 5.2 is commutative. Then $I^2 = 1$ if and only if $c_1 + c_2 = 0$.*

Corollary 5.5. *The cio $Q_iK : S \to S$ has order 2.*

Proof: The group in Corollary 5.3 is commutative and

$$\begin{aligned} c_1 + c_2 &= x_2 - x_1 - i + x_1 - x_2 + i \\ &= (x_2 - x_2) + (x_1 - x_1) + (i - i) \\ &= 0, \end{aligned}$$

so the cio Q_iK satisfies $(Q_iK)^2 = 1$ by the previous Lemma. □

The next theorem from Kochavi tells us the structure of the groups considered above.

Theorem 5.6. *Let G be the group of operations on S generated by a cio I and the transposition operation T_1. Suppose G is commutative. Let $f_1(z) = 2z + c_1$ and $f_2(z) = 2z + c_2$ be the indexing function pieces for I. Then the following statements hold.*

1. *If $c_1 + c_2$ is even, then $G \cong \mathbb{Z}_{12} \bigoplus \mathbb{Z}_2$.*
2. *If $c_1 + c_2$ is odd, then $G \cong \mathbb{Z}_{24}$.*

Corollary 5.7. *The group of operations on S generated by the cio Q_iK and the transposition T_1 is isomorphic to $\mathbb{Z}_{12} \bigoplus \mathbb{Z}_2$.*

Proof: Recall that $c_1 + c_2 = 0$ from the proof of Corollary 5.5. □

Theorem 5.8. *The group of operations on S generated by the cio Q_iK and the transposition T_1 acts simply transitively on S.*[35]

Proof: Let H denote the group. One can see that H acts transitively on S. To see that it acts simply transitively, suppose $O_1, O_2 \in H$ and $O_1Y = O_2Y$ for some $Y \in S$. Then $O_2^{-1}O_1 \in H_Y$ where H_Y denotes the stabilizer of Y in H. The orbit stabilizer theorem implies

$$|H|/|H_Y| = |\text{orbit of } Y|$$

$$24/|H_Y| = 24$$

and thus $|H_Y| = 1$. Therefore $O_2^{-1}O_1 = 1_S$ and $O_2 = O_1$, which implies simple transitivity. □

This concludes our literature review.

[35]As mentioned in Kochavi 1998, 307, such a group is essential to Lewin's analysis of Stockhausen; Lewin's recombined group has J as the cio.

6. Hindemith, *Ludus Tonalis*, Fugue in E

This section illustrates some of the ideas presented thus far through an exploratory analysis of Hindemith, *Ludus Tonalis*, Fugue in E. We take as our generating pitch-class segment $X = \langle 2, 11, 4, 9\rangle = \langle D, B, E, A\rangle$, which is a motive in the subject. The motive is stated and then sequentially repeated three times in measures 1-4 of Figure 1. Table 1 below lists all the transposed and inverted forms of X, which are designated by P_n and p_n respectively.[36] As in Section 3, the set of all transposed and inverted forms of the pitch-class segment X is denoted S.

The pitch-class *set* $\{2, 11, 4, 9\}$ is I-symmetric, and thus a neo-Riemannian-type group will not act simply transitively on its set of transposed and inverted forms.[37] Despite this, the pitch-class *segment* $\langle 2, 11, 4, 9\rangle$ supports a simply transitive group action on its set of transposed and inverted forms because $\langle 2, 11, 4, 9\rangle$ satisfies our condition at the beginning of Section 3 (the distinct pitches 2 and 11 span an interval different from a tritone). Our theory's use of pitch-class *segments* (rather than pitch-class *sets*) is desirable since it captures a musical feature that may otherwise be overlooked. In this piece Hindemith treats an inverted form of the pitch-class segment as distinct from the transposed form that has the same pitch content, so pitch-class ordering matters. For instance, the $\langle B, D, A, E\rangle$ motive in measure 35 is presented as a transformation of the initial $\langle D, B, E, A\rangle$ motive.[38] As we see in Figure 2, $\langle B, D, A, E\rangle$ is contained within the I_{11} transformation of the fugue subject: $p_{11} = I_{11}P_0$. Within the I_{11} transformation, $\langle B, D, A, E\rangle$ is a Q_{10} transposition, that is, $\langle B, D, A, E\rangle = I_{11}Q_{10}\langle D, B, E, A\rangle$.[39] So in the context of this piece it is useful to consider motives as distinct whenever they have different orderings, even if they have the same pitch-class content.

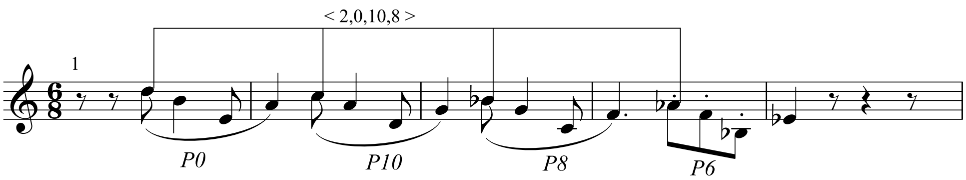

Figure 1.

[36]Lewin 2003 notes that p_0 traditionally is reserved for "that inverted form that is most characteristically paired with P_0," although often times it is not clear which inverted form should be called p_0, as is the case in Schoenberg, op. 23, no. 3. Our choice of p_0 was dictated strictly by the notational convenience of the equations $I_0P_0 = p_0$ and $I_nP_0 = p_n$.

[37]This pitch-class set is I-symmetric because $I_1\{2, 11, 4, 9\} = \{2, 11, 4, 9\}$. As a result of this symmetry, the set of all transposed and inverted forms of the pitch-class set $\{2, 11, 4, 9\}$ has only twelve elements. If a group acts simply transitively on a set, then the group and the set have the same number of elements. Therefore, it is impossible for a neo-Riemannian group of order 24 to act simply transitively on this 12 element set.

[38]$I_1\langle D, B, E, A\rangle = \langle B, D, A, E\rangle$.

[39]The dual group makes its first appearance here, as we will soon see.

Prime Forms	
P_0	$\langle 2, 11, 4, 9\rangle$
P_1	$\langle 3, 0, 5, 10\rangle$
P_2	$\langle 4, 1, 6, 11\rangle$
P_3	$\langle 5, 2, 7, 0\rangle$
P_4	$\langle 6, 3, 8, 1\rangle$
P_5	$\langle 7, 4, 9, 2\rangle$
P_6	$\langle 8, 5, 10, 3\rangle$
P_7	$\langle 9, 6, 11, 4\rangle$
P_8	$\langle 10, 7, 0, 5\rangle$
P_9	$\langle 11, 8, 1, 6\rangle$
P_{10}	$\langle 0, 9, 2, 7\rangle$
P_{11}	$\langle 1, 10, 3, 8\rangle$

Inverted Forms	
p_0	$\langle 10, 1, 8, 3\rangle$
p_1	$\langle 11, 2, 9, 4\rangle$
p_2	$\langle 0, 3, 10, 5\rangle$
p_3	$\langle 1, 4, 11, 6\rangle$
p_4	$\langle 2, 5, 0, 7\rangle$
p_5	$\langle 3, 6, 1, 8\rangle$
p_6	$\langle 4, 7, 2, 9\rangle$
p_7	$\langle 5, 8, 3, 10\rangle$
p_8	$\langle 6, 9, 4, 11\rangle$
p_9	$\langle 7, 10, 5, 0\rangle$
p_{10}	$\langle 8, 11, 6, 1\rangle$
p_{11}	$\langle 9, 0, 7, 2\rangle$

Table 1. The Elements of S.

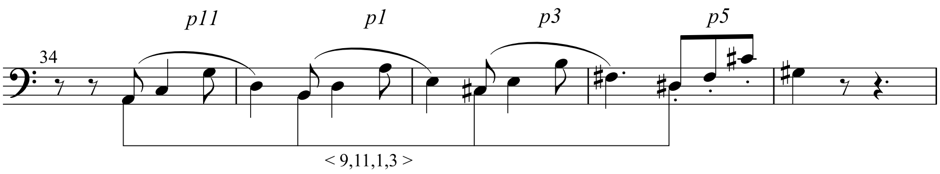

Figure 2.

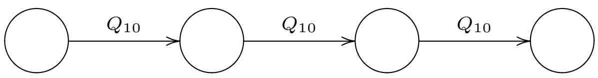

Figure 3.

Next we use transformational graphs and networks[40] to elucidate the role of the generalized contextual group, which is the dual group to the GIS of transposition and inversion on S. The transformational graph in Figure 3 shows the transformational structure of the subject statement in measures 1-4 of Figure 1. If we fill in the first node with the motive $P_0 = \langle D, B, E, A\rangle$, then we obtain the rest of the subject by successively applying the element Q_{10}, which is the element of the contextual group that transposes prime forms by 10 and inverted forms by $-10 = 2$.

The relationship between the subject entry in measures 1-4 of Figure 1 and the entry in measures 34-37 of Figure 2 is contained in the transformational network of Figure 4. Measure numbers are shown to the upper right of nodes. The horizontal transformations Q_{10} are from the generalized contextual group $COMM$ whereas the vertical transformations are from the T/I group $SIMP$, which is the group of transpositions of our GIS. This is a well-formed network[41] because of the dual relationship between $COMM$ and $SIMP$. In other words, if we follow any two

[40]Roughly speaking, a graph is a network without node content. See Lewin 1987, 195-196.

[41]A category theorist would say that the diagram *commutes*.

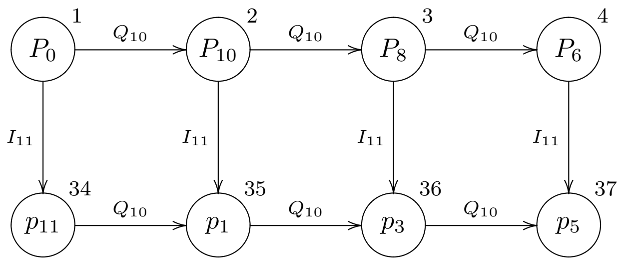

FIGURE 4.

paths with the same initial point and the same terminal point then the result will be the same because each element of $COMM$ commutes with each element of $SIMP$. More specifically $Q_{10}I_{11}$ is equal to $I_{11}Q_{10}$.[42] This is what Lewin refers to as a product network.[43] Its underlying graph is a product of the graph in Figure 3 and the graph consisting of one arrow I_{11} with two nodes. One can say generally that any product of a well-formed graph in $COMM$ with a well-formed graph in $SIMP$ will itself be well formed because of the commutativity. Notice that the contextual group makes this network possible: if one tried to use T_{10} instead of Q_{10} the relationship in Figure 4 would be lost.

The perspective given by these dual groups highlights the occurrence of the same pattern at different levels in the work, that is, occurrences of "self similarity." Figure 5 and Figure 6 show the replication of the pitch-class segment's internal structure in the transformations that span the work as a whole. In Figure 5 we see the elements of the pitch-class segment P_5 related by operations in $SIMP$. Although the objects in Figure 5 are individual pitch classes and not pitch-class segments, we may still apply operations from $SIMP$ to them since $SIMP$ operations in this situation are the operations of the T/I group and therefore are well defined on individual pitch classes. The interpretation in Figure 6 shows a product network obtained from the underlying graphs of Figures 3 and 5. This product network relates subject statements in four sections of the work. As in Figure 4, in Figure 6 the horizontal transformations Q_{10} are from the generalized contextual group $COMM$ whereas the vertical transformations are from the T/I group $SIMP$. The separated left column in Figure 6, though not part of the product network, is shown to illustrate that the rows of the product network are related to each other in the same way that the pitch classes of Figure 5 are; this is the self similarity mentioned earlier. The relevant subject entries from the fugue are also recorded below in Figures 7-10.

[42]Networks formed from the T/I group that do not involve the dual group are not always well formed. For problems in forming networks using only the T/I group see Lewin 1995, 107-109.

[43]For a definition of *product network* see Lewin 1995, 108.

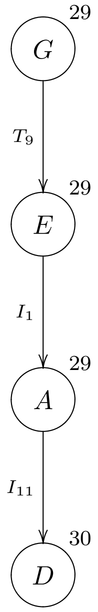

FIGURE 5.

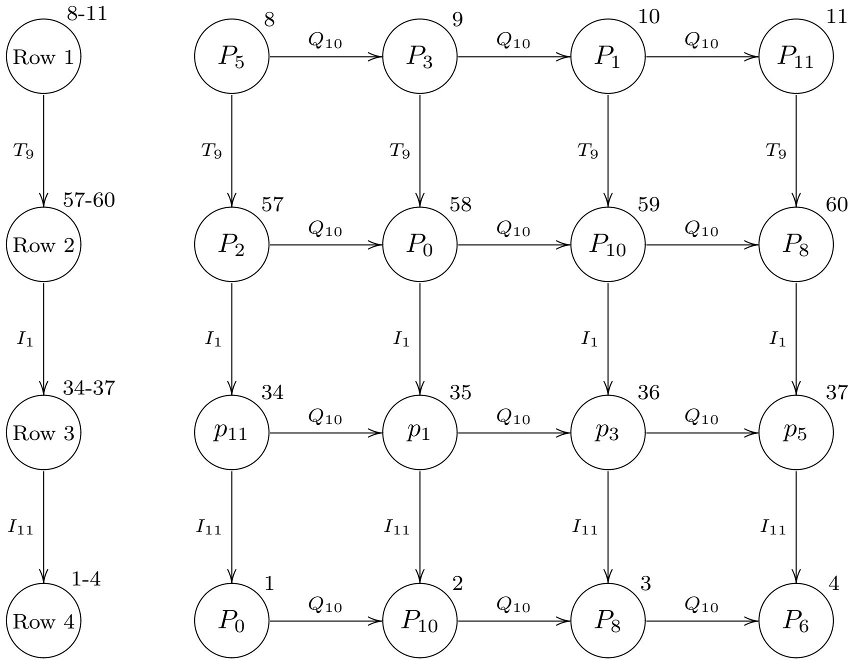

FIGURE 6.

Lewin notes that by choosing different groups, one can obtain different analyses.[44] What results if instead of choosing a different group, one holds the group constant and chooses a different space of musical objects? In the case of the fugue,

[44]Choices of canonical groups are discussed in Lewin 1987, 104-122 and 150-153.

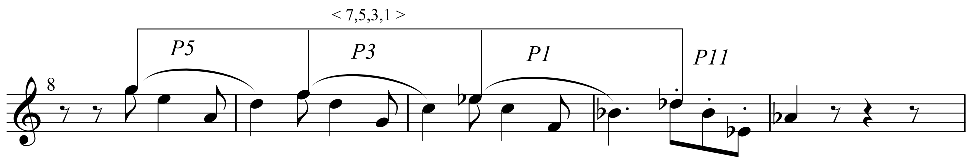

FIGURE 7.

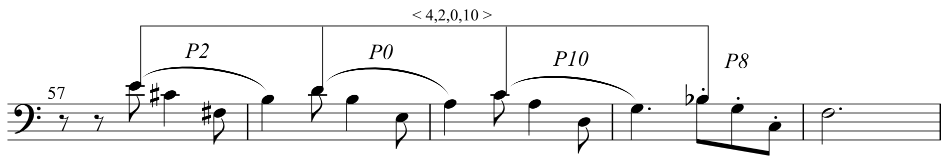

FIGURE 8.

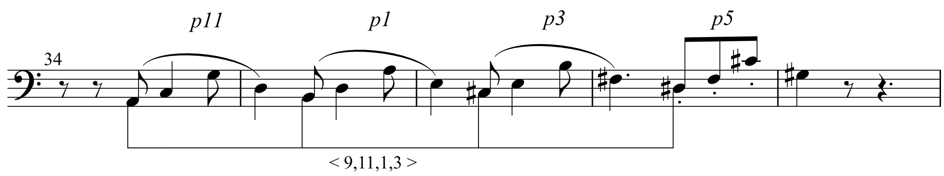

FIGURE 9.

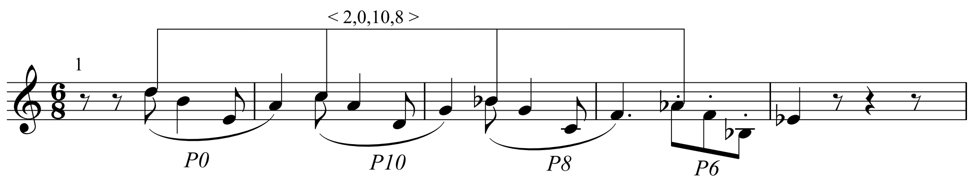

FIGURE 10.

we can demonstrate another kind of self similarity by doing exactly this, that is, by comparing the actions of a given group on two different musical spaces: S, the space of the forms of pitch-class segment $\langle D, B, E, A\rangle$ on one hand, and S', the space of the forms of pitch-class segment $\langle D, C, B\flat, A\flat\rangle$ on the other. This analytical move allows us to notice if transformational patterns are shared by distinct musical objects. Specifically, we have the T/I group acting simply transitively on two musical spaces S and S'.[45]

Table 2 lists the members of S', the transposed and inverted forms of the pitch-class segment $X' = \langle D, C, B\flat, A\flat\rangle = \langle 2, 0, 10, 8\rangle$. The S' member $\langle D, C, B\flat, A\flat\rangle$ is obtained by tracking the first pitch of each motive in the subject's initial appearance

[45]In the discussion of Figure 5 and Figure 6 in the previous paragraph we also had the T/I group acting on two musical spaces. The spaces are $\mathbb{Z}_{12}$, ordinary pitch-class space, and S, the forms of the pitch-class segment $\langle D, B, E, A\rangle$. However, the T/I group did *not* act simply transitively on the set of 12 pitch classes in Figure 5, so the current example is more sophisticated.

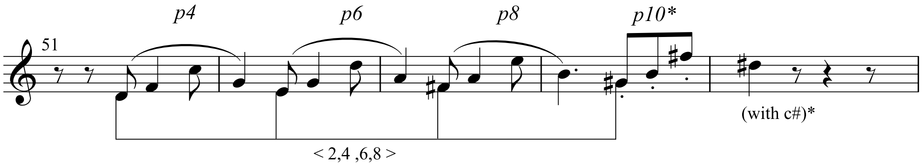

FIGURE 11.

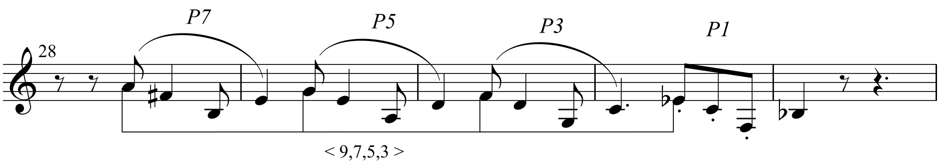

FIGURE 12.

(Figure 1). As discussed already, S denotes the set of the transposed and inverted forms of the pitch-class segment $X = \langle D, B, E, A\rangle = \langle 2, 11, 4, 9\rangle$ in Table 1.

Prime Forms		Inverted Forms	
P'_0	$\langle 2, 0, 10, 8\rangle$	p'_0	$\langle 10, 0, 2, 4\rangle$
P'_1	$\langle 3, 1, 11, 9\rangle$	p'_1	$\langle 11, 1, 3, 5\rangle$
P'_2	$\langle 4, 2, 0, 10\rangle$	p'_2	$\langle 0, 2, 4, 6\rangle$
P'_3	$\langle 5, 3, 1, 11\rangle$	p'_3	$\langle 1, 3, 5, 7\rangle$
P'_4	$\langle 6, 4, 2, 0\rangle$	p'_4	$\langle 2, 4, 6, 8\rangle$
P'_5	$\langle 7, 5, 3, 1\rangle$	p'_5	$\langle 3, 5, 7, 9\rangle$
P'_6	$\langle 8, 6, 4, 2\rangle$	p'_6	$\langle 4, 6, 8, 10\rangle$
P'_7	$\langle 9, 7, 5, 3\rangle$	p'_7	$\langle 5, 7, 9, 11\rangle$
P'_8	$\langle 10, 8, 6, 4\rangle$	p'_8	$\langle 6, 8, 10, 0\rangle$
P'_9	$\langle 11, 9, 7, 5\rangle$	p'_9	$\langle 7, 9, 11, 1\rangle$
P'_{10}	$\langle 0, 10, 8, 6\rangle$	p'_{10}	$\langle 8, 10, 0, 2\rangle$
P'_{11}	$\langle 1, 11, 9, 7\rangle$	p'_{11}	$\langle 9, 11, 1, 3\rangle$

TABLE 2. The Elements of S'.

The pitch-class segment $X' = \langle D, C, B\flat, A\flat\rangle$ satisfies our condition at the beginning of Section 3 (D and C span an interval other than a tritone), so the GIS consisting of the T/I group acting simply transitively on S' has a generalized contextual group as its dual. The contextual inversions L and L' are elements of the generalized contextual groups for S and S', respectively. We define the contextual inversions $L : S \to S$ and $L' : S' \to S'$ as those operations which map an input segment to that inverted form of the segment which has the same first note.[46] For example, using Tables 1 and 2, we see

$$L\langle \mathbf{2}, 11, 4, 9\rangle = \langle \mathbf{2}, 5, 0, 7\rangle,$$

[46]This L is not the same as the L in Lewin 2003.

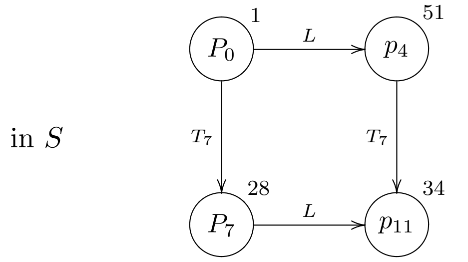

FIGURE 13.

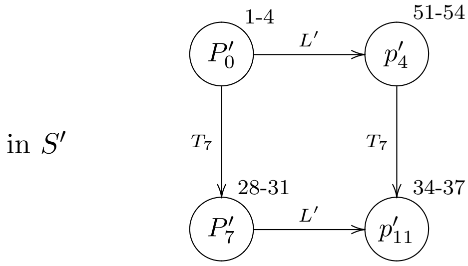

FIGURE 14.

$$L\langle \mathbf{9}, 6, 11, 4\rangle = \langle \mathbf{9}, 0, 7, 2\rangle,$$
$$L'\langle \mathbf{2}, 0, 10, \mathbf{8}\rangle = \langle \mathbf{2}, 4, 6, \mathbf{8}\rangle,$$
$$L'\langle \mathbf{9}, 7, 5, \mathbf{3}\rangle = \langle \mathbf{9}, 11, 1, \mathbf{3}\rangle.$$

Here, first notes 2 and 9 (shown in boldface) are held fixed under both L and L'.

In the example, the last notes 8 and 3 (shown in boldface) also are held fixed under L'; we could just as well describe $L' : S' \to S'$ as the operation which maps a segment to its inverted form which has the same *last* entry. This is because the first and last entry of every member of S' differ by a tritone. The analogous description does not work for $L : S \to S$. Corollary 4.4 tells us that the definition of the generalized contextual group does not depend on the choice of pair of notes exchanged by the generating inversion operation, so we automatically know that L and L', as defined above, are members of their respective generalized contextual groups. We can verify the membership of L and L' in their respective groups by setting $k = \ell = 1$ in the proof of the corollary.

Both S and S' are prominent in the piece. Instances of members of S', the whole-tone tetrachord, are shown with beams in Figures 1-12, and instances of members of S, the fugue subject's head motive, are labeled with P and p in Figures 1-12. As indicated in the figures, a form of the whole-tone tetrachord X' (i.e. an element of S') is articulated at the "middle ground" of each subject statement. Figure 13 and Figure 14 display similar transformational patterns in distinct GIS's and their duals.

Figure 15 reconfigures Figure 14 in the correct temporal sequence to illustrate a symmetry that spans the work. We also see that pitch classes $0, \ldots, 11$ appear in the network: the fugue exhausts all pitch classes as initial tones of the head motive. The first and last pitch classes are highlighted because they are preserved by L'.

We now consider episodic material which contains neither subject nor answer. Here we apply the perspective of generalized contextual groups to unordered sets

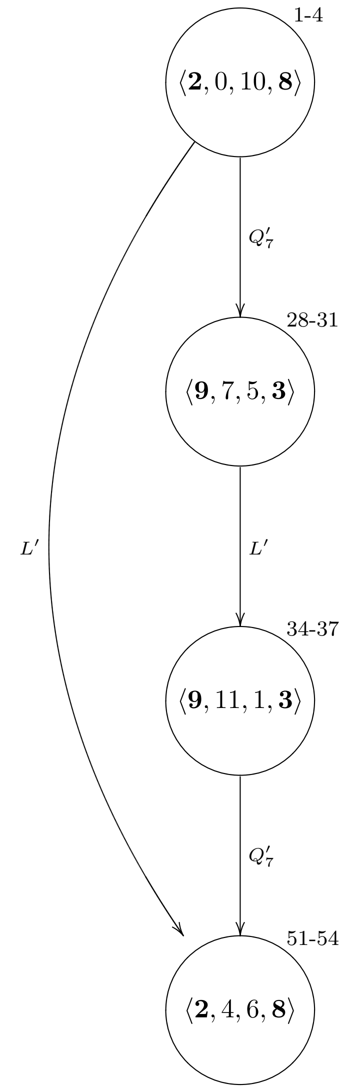

Figure 15.

which exhibit inversional symmetry. In this application, the analyst will have more than one choice in selecting the content of nodes in a network. For instance, P_0 is $\langle 2, 11, 4, 9\rangle$ whereas p_1 is $\langle 11, 2, 9, 4\rangle$. The two segments project the same pitch-class content and therefore may both be thought of as corresponding to the first tetrachord in the upper voice of Figure 16, pitch-class set $\{2, 4, 9, 11\}$.

Let $J : S \to S$ denote the contextual inversion that switches the second and fourth pitch classes of the input pitch-class segment. Let $K : S \to S$ denote the contextual inversion that holds the last pitch fixed on the input pitch-class segment. The left network of Figure 18 analyzes the episodic material in Figure 16 and Figure 17 in terms of J-inversion. The right network is an analysis of the same passage in terms of K-inversion. The episode's pairing of P_0/p_1 with p_8/P_7, as shown in the top row of Figure 18, recurs in a note-against-note setting in the coda (Figure 19).

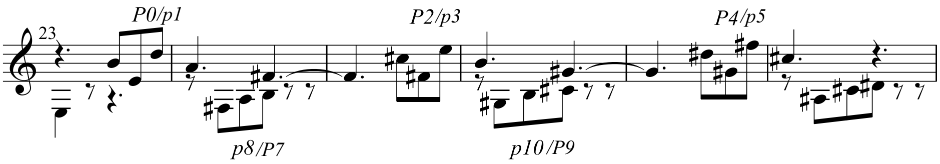

FIGURE 16.

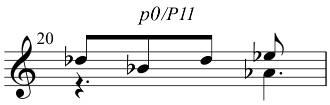

FIGURE 17.

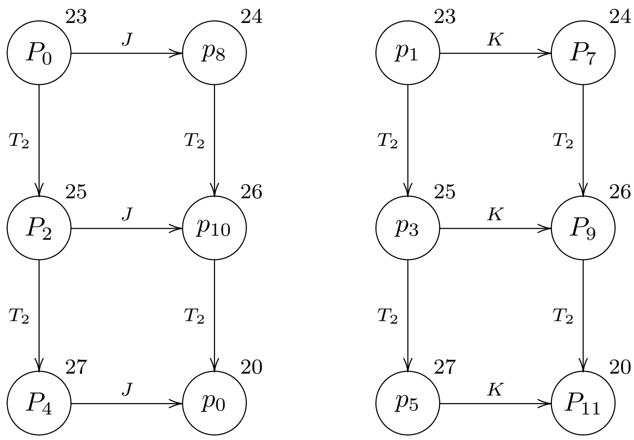

FIGURE 18.

Which network is preferable, the one that features J-inversion or the one that features K-inversion? The K-inversion network is perhaps more relevant because K resembles L' in preserving the final pitch of the input segment, that is, K and L' are the same group operation but in different GIS's, each with a different space.

The segmental approach developed in this paper defines operations applicable to the set material in the episodes and coda. By using a space of pitch-class segments to analyze unordered sets, we arrived at a description of the fugue episode as a well-formed network involving the dual groups: the T/I group on one axis and the contextual group on the other. While this is only a brief analytical illustration, it does suggest one way to extend the theory of contextual groups acting on segments to include the analysis of unordered sets, even those with inversional symmetry.

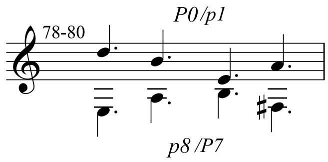

FIGURE 19.

7. More Generalized Contextual Groups

So far we have only considered pitch-class segments with entries in $\mathbb{Z}_{12}$ and the corresponding generalized contextual groups. There is also a generalized contextual group for any $\mathbb{Z}_m$.[47] This justifies the term "generalized." The transpositions and inversions *mod m* below are defined analogously to the transpositions and inversions *mod* 12.

Theorem 7.1. *Let $x_1, \ldots, x_n \in \mathbb{Z}_m$ and suppose that there exist x_q, x_r in the list such that $2(x_q - x_r) \neq 0$. Let S be the family of $2m$ pitch-class segments that are obtained by transposing and inverting the pitch-class segment $X = \langle x_1, \ldots, x_n \rangle$. Then the $2m$ transpositions and inversions act simply transitively on S.*

Fix $1 \leq k, \ell \leq n$. Define $K(Y) := I_{y_k + y_\ell}(Y)$ and $Q_i(Y)$ as before for $Y \in S$. Then K and Q_1 generate the commuting group for the T/I group of order $2m$. This commuting group is also referred to as a generalized contextual group. This generalized contextual group is dihedral of order $2m$. It is also the group of interval-preserving operations for the GIS associated to the simply transitive group action of the mod m T/I group.

Proof: Let S be the set of transposed and inverted forms of X. We claim that S has $2m$ elements. It suffices to consider the case $X = \langle x_1, x_2 \rangle$ with $2(x_1 - x_2) \neq 0$. Let S_T denote the set of transposed forms of X and let S_I denote the set of inverted forms of X. Then $S = S_T \cup S_I$. We have either $S_T \cap S_I = \emptyset$ or $S_T = S_I$. Suppose $S_I = S_T$. Then $\langle x_1, x_2 \rangle$ is in both sets and there exists some $0 \leq i \leq m-1$ such that

$$x_1 = -x_1 + i$$
$$x_2 = -x_2 + i$$

This implies $2x_1 = i = 2x_2$ and $2(x_1 - x_2) = 0$, which contradicts the assumption $2(x_1 - x_2) \neq 0$. Thus $S_T \neq S_I$ and $S_T \cap S_I = \emptyset$. Each of S_T and S_I has m elements, and so S has $2m$ elements.

We claim that the group of *mod m* transpositions and inversions acts simply transitively on S. Let H denote the *mod m* T/I group, which acts transitively on S because of the way S is defined. Then for $Y \in S$, the orbit of Y is all of S. Hence the orbit stabilizer theorem implies

$$|H|/|H_Y| = |\text{orbit of } Y|$$

$$2m/|H_Y| = 2m$$

and thus $|H_Y| = 1$. The simple transitivity now follows as in the proof of Theorem 5.8.

[47]A similar observation appears in Hook 2002, 106.

The rest of proof becomes the same as for $m = 12$ if one replaces 12 with m and 24 with $2m$ in Sections 3 and 4. □

Note that the choice of k and ℓ between 1 and n is irrelevant, just as in the $\mathbb{Z}_{12}$ case. For $m = 12$, the condition on the existence of x_q, x_r such that $2(x_q - x_r) \neq 0$ is precisely the requirement that there are two distinct pitch classes which span an interval other than a tritone.

After generalizing Kochavi's results to the *mod m* universe, one can prove the following structure theorem about recombined groups.

Theorem 7.2. *The group of operations on S generated by the mod m cio Q_iK of order 2 and the mod m transposition T_1 is isomorphic to $\mathbb{Z}_m \bigoplus \mathbb{Z}_2$. Furthermore, this group of operations on S acts simply transitively on S.*

8. A Musical Example of a Mod 3 System

Theorem 7.1 permits us to generalize beyond the traditional $\mathbb{Z}_{12}$ system. We now give an example from $\mathbb{Z}_3$ of a six-element system for triad voicings. Here we have $m = 3$, $X = \langle 0, 1, 2\rangle$, $q = k = 1$, and $r = \ell = 2$ in the notation of Theorem 7.1, so $2(0 - 1) = -2 = 1 \neq 0$ and the theorem applies.

Table 3 lists the transposed and inverted forms of X in the musical space S. There are $2m = 2 \cdot 3 = 6$ elements of S. The segments are listed next to their corresponding spacing types. One should keep in mind that we are not looking at pitch-sets but chord spacings. Accordingly, the numbers do not refer to specific pitches. Segment numbers give names for chord spacings; since spacing is the only property at issue, in this system a given chord may contain any number of pitches. Prime forms represent closed triad voicings and inverted forms represent open triad voicings.[48]

Table 4 shows some contextual operations evaluated on X. The operation K exchanges the first two elements of the segment, so, for instance, $K\langle 0, 1, 2\rangle = \langle 1, 0, 2\rangle$. The operation Q_1 transposes prime forms by 1 *mod* 3 and inverted forms by -1 *mod* 3, so $Q_1\langle 0, 1, 2\rangle = \langle 1, 2, 0\rangle$ while $Q_1\langle 1, 0, 2\rangle = \langle 0, 2, 1\rangle$. The operations T_n and I_n are defined in the usual ways: $I_n(x) = -x + n$ (*mod* 3), and $T_n(x) = x + n$ (*mod* 3). Applying the operations to pitch-class segments is componentwise, so $I_0\langle 1, 0, 2\rangle = \langle -1, -0, -2\rangle = \langle 2, 0, 1\rangle$. Segment labels are attached to chords A through F in the chorale phrase of Figure 20. Theorem 7.1 tells us that T_1 and K commute, a fact which permits us to construct the dual-group network in Figure 21. As in previous figures, the horizontal operations are from the generalized contextual group $COMM$ and the vertical transformations are from the T/I group $SIMP$.

[48]Here "open position" denotes chord spacings in which the notes above the bass are not as close together as possible. Although the term is usually applied to root-position triads, it may be applied to inversions, as in Leonard Ratner, *Harmony: Structure and Style* (New York: McGraw Hill, 1962), 30.

Prime Forms	Inverted Forms
root position, closed= $\langle 0,1,2 \rangle$	$\langle 0,2,1 \rangle$ = root position, open
first inversion, closed = $\langle 1,2,0 \rangle$	$\langle 1,0,2 \rangle$ = first inversion, open
second inversion, closed= $\langle 2,0,1 \rangle$	$\langle 2,1,0 \rangle$ = second inversion, open

Table 3. The Elements of S.

Prime Forms	Inverted Forms
$Q_0(X) = \langle 0,1,2 \rangle$	$\langle 0,2,1 \rangle = Q_1K(X)$
$Q_1(X) = \langle 1,2,0 \rangle$	$\langle 1,0,2 \rangle = K(X)$
$Q_2(X) = \langle 2,0,1 \rangle$	$\langle 2,1,0 \rangle = Q_2K(X)$

Table 4. Some Contextual Operations Evaluated on X.

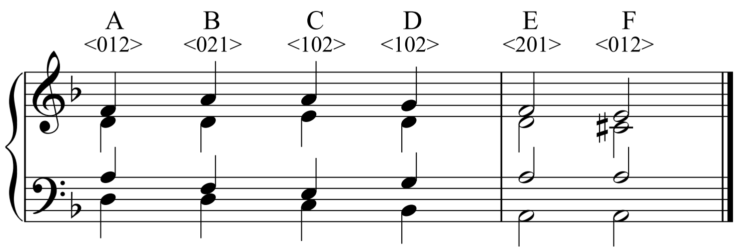

Figure 20.

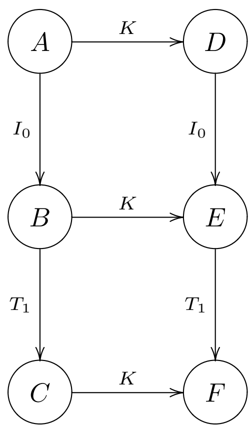

Figure 21.

9. Conclusion

Recent research on the space of 24 consonant triads has found that the neo-Riemannian L/R group is the dual of the T/I group and that the perspective of dual groups is fruitful for music analysis. We have extended this work by showing how analysis with dual groups can be pursued not just with consonant triads, but with any pitch-class segment which contains at least one interval other than a tritone or unison. Our theory applies even to segments whose underlying pitch-class set is inversionally symmetrical. By reformulating the tritone condition in Section 7, we extended the theory to apply to all pitch-class universes of finite size. Specifically, if X is a pitch-class segment satisfying the condition, then the T/I group acts simply transitively on the family S of all transposed and inverted forms of X, and its dual group is precisely the generalized contextual group. In any $\mathbb{Z}_m$ universe, the generalized contextual group gives rise to an action of the dihedral group of order $2m$. The recombined group generated by a contextual inversion Q_iK and the transposition T_1 is isomorphic to $\mathbb{Z}_m \bigoplus \mathbb{Z}_2$ and acts simply transitively on S.

10. Appendix

We present here a proof of Theorem 2.1. Let the notation be as in Theorem 2.1.
Proof:

Let $PRES$ denote the group of interval-preserving operations for the GIS.

We claim that $COMM \subseteq PRES$. Let $f : S \to S$ be an element of $COMM$. We must show that f preserves intervals. To this end, let $x, y \in S$ and let $i := int(x, y)$. Then

$$\begin{aligned} T_i(x) &= y \\ f(T_i(x)) &= f(y) \\ T_i(f(x)) &= f(y), \end{aligned}$$

which implies $int(f(x), f(y)) = i$ by the definition of GIS transposition. Hence $int(f(x), f(y)) = int(x, y)$ and f preserves intervals, so $f \in PRES$. Therefore $COMM \subseteq PRES$.

We claim that $PRES \subseteq COMM$. Let $f : S \to S$ be an element of $PRES$. Let $x \in S$ and $i \in I$. We must show that $f(T_i(x)) = T_i(f(x))$. Since f is interval-preserving, we have $i = int(x, T_i(x)) = int(f(x), f(T_i(x))$. By the definition of GIS transposition we also have $i = int(f(x), T_i(f(x)))$. Combining the previous two expressions we see $int(f(x), f(T_i(x))) = i = int(f(x), T_i(f(x)))$, which implies $f(T_i(x)) = T_i(f(x))$ by the uniqueness requirement on the interval function in the definition of GIS in Section 2. Hence T_i and f commute for all $i \in I$. Therefore $f \in COMM$ and $PRES \subseteq COMM$.

We conclude that $COMM = PRES$, that is, the group of operations which commute with the transpositions is precisely the group of interval-preserving operations for the GIS. □

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