MTO 0.3 Examples: Lindley and Turner-Smith, Mathematical Models of Scales

(Note: audio, video, and other interactive examples are only available online)

http://www.mtosmt.org/issues/mto.93.0.3/mto.93.0.3.lindley_turner-smith.php

Figure 1. A sequence of constructions abstracted from the pitch continuum

a) a positive-number line for pitch frequencies

b) a number line for logarithms to base 2 of the frequencies

c) equivalence classes of points mod 1 on this number-line (flogs)

d) pitch-class relations between these equivalence classes (also flogs)

e) two kinds of generators for groups of pitch-class relations:
   equal-division (1/n converted to a flog) or harmonic (see below)

f) the pairs, (set, Abelian group), which ensue from these generators

g) equivalence-class neighborhoods around every point-mod-1 in this set

h) "ideal systems", whereby every system comprises a pair:
   a finite set of non-overlapping neighborhoods, operated upon by
   a subset of one of our groups (that is, by a "halfgroup")

i) unlimited scales (repeating in every octave indefinitely)

j) scales with a highest and lowest note

Figure 2. Three pitches in one pitch class

<table>
<thead>
<tr>
<th>pitches:</th>
<th>------</th>
<th>+ +---</th>
<th>------</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequencies:</td>
<td>+ - +</td>
<td>+ + + +</td>
<td>+ - +</td>
</tr>
<tr>
<td>our model:</td>
<td>+ - +</td>
<td>+ + + +</td>
<td>+ - +</td>
</tr>
</tbody>
</table>
GIF 1. How to reckon in flogs rather than logs to base 2

\[
\begin{align*}
\text{instead of:} & \quad \begin{array}{c}
0.585 + 0.585 \\
= 1.170
\end{array} \\
\text{we have:} & \quad \begin{array}{c}
0.585 + 0.585 \\
= 1.170
\end{array}
\end{align*}
\]

Figure 3. Three notes in one pitch class

\[\begin{align*}
\text{octave} \quad -\quad (\quad +\quad ) \quad -\quad (\quad +\quad ) \quad -\quad (\quad +\quad ) \quad \text{octave}
\end{align*}\]

\[\begin{align*}
\text{note} \quad -\quad (\quad +\quad ) \quad -\quad (\quad +\quad ) \quad -\quad (\quad +\quad ) \quad \text{note}
\end{align*}\]

GIF 2. Some types of algebraic structure

Figure 4. Our notation for the generating harmonic pitch-class relations

"I" = flog 2 (= 0, thus the identity element)
"V" = flog 3 plus or minus a much smaller flog (t^V^)
"III" = flog 5 plus or minus a much smaller flog (t^III^)
"VII" = flog 7 plus or minus a much smaller flog (t^VII^)
Figure 5. A rough classification of orders of intervallic magnitude

(ca. 10 octaves - range of hearing)
(ca. 1 octave - difference between men's and women's voices)
10ths of an octave - melodic steps and leaps
100ths of an octave - out-of-tune-ness (Such a t would mar a consonance.)
1000ths of an octave - tempering
1/10 000-octave - musically insignificant

GIF 3. Reckoning the size of a chromatic semitone

\[
\frac{5}{4} \times \frac{5}{6} = \frac{25}{24} \\
\text{log} \left( \frac{25}{24} \right) / \log 2 = 0.06
\]

Figure 6. A table showing how to find the most feasible equations \( nV = III \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m(\log 3) )</th>
<th>( \log 5 = s^n )</th>
<th>( m^n )</th>
<th>( s^n / (m^n + 1) )</th>
<th>( T^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-( \log 3 )</td>
<td>-( \log 5 )</td>
<td>.0931</td>
<td>1</td>
<td>.0466</td>
</tr>
<tr>
<td>2</td>
<td>3 ( \log 3 )</td>
<td>+ ( \log 5 )</td>
<td>.0768</td>
<td>3</td>
<td>.0192</td>
</tr>
<tr>
<td>3</td>
<td>4 ( \log 3 )</td>
<td>- ( \log 5 )</td>
<td>.0179</td>
<td>4</td>
<td>.0036</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>\rightarrow 4V = III</td>
</tr>
<tr>
<td>4</td>
<td>8 ( \log 3 )</td>
<td>+ ( \log 5 )</td>
<td>.0016</td>
<td>8</td>
<td>.0002</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>\rightarrow -8V = III</td>
</tr>
<tr>
<td>5</td>
<td>45 ( \log 3 )</td>
<td>- ( \log 5 )</td>
<td>.0014</td>
<td>45</td>
<td>.00003</td>
</tr>
</tbody>
</table>

Figure 7. Branches in our system-tree
a) Among systems:
   harmonic
   equal-division

b) Among harmonic systems:
   1-dimensional (the generators are \{I, V\})
   2-dimensional (the generators are \{I, V, III\})
   3-dimensional (the generators are \{I, V, III, VII\})

c) Also among harmonic systems:
   coherent (comprising one chain of V's)
   not coherent

d) Among coherent systems:
   untempered (all \( t = 0 \))
   tempered

e) Among temperaments:
   regular (all \( t^V \) equal, all \( t^{III} \) equal, etc.)
   semi-regular (all \( t^V \) equal, but not all \( t^{III} \) or \( t^{VII} \))
   irregular (\( t^V \) and hence \( t^{III} \) & \( t^{VII} \) varying)

f) Among regular, two- (and sometimes three-) dimensional temperaments:
   MT (wherein \( 4V = III \))
   QP (wherein \(-8V = III \))

g) Conjunctions of MT and QP:
   ET^1^ (12V = I; there is no III and no VII)
   ET^2^ (12V = I; 4V = III = -8V; there is no VII)
   ET^3^ (12V = I; 4V = III = -8V; -2V = VII = 10V)

h) Apart from ET, some musically good possibilities for MT or for QP:
   meantone temperaments
   quasi-Pythagorean temperaments

i) Physically equivalent equal-division systems,
   for instance, the mere division of the octave into 12 equal parts
   (a system which is physically equivalent to an ET but has no diatonic
   or chromatic semitones)

j) Certain families of such equal-division systems:
   F^1^ (characterized by equivalence to MT systems)
   F^2^ (characterized by equivalence to QP systems)

k) A kind of temperament, CT, which includes ET and some irregular temperaments
   that approximate to an ET within a certain "margin of equivalence"
   and meet certain other specifications

l) Some historically important kinds of CT:
   JSB
   *temperament ordinaire* (with an accent over the second "e")

m) A "semi-regular" temperament that is physically equivalent to a
   quasi-Pythagorean temperament but has two different values for III
Figure 8. Diagram of a meantone system with 14 pitch classes

("--" between two note-names means there is a V relation between the two pitch classes; "/" means there is a III relation between them. Imagine the "--"s spiraling up on a cylinder from Ab to D#

\[
\begin{align*}
C\# & \quad G\# & \quad D\# \\
/ & \quad / & \quad / \\
A & \quad E & \quad B & \quad F\# & \quad (C\#) \\
/ & \quad / & \quad / & \quad / \\
F & \quad C & \quad G & \quad D & \quad (A) \\
/ & \quad / & \quad / & \quad / \\
Ab & \quad Eb & \quad Bb & \quad (F)
\end{align*}
\]

GIF 4. Schütz, "Die so ihr den Herren fürchtet" (SWV 164)

(a) A diagram showing when the first and last use of each pitch class occurs

(b-d) Three excerpts showing some of these uses
GIF 5. A 15th-century quasi-Pythagorean system and some evidence of its use; music in which triads with an explicit sharp are especially salient
b. Landini, "O fanciulla giulia" (ballata),
first ending of the second section

![Musical notation image]

---

c. Matteo da Perugia, "A qui fortune" (rondeau),
end of the first section

![Musical notation image]

---

d. Matteo da Perugia, "Le grant desir" (ballade),
end of the first section

![Musical notation image]

---

e. End of an organ verse from the Faenza Codex

![Musical notation image]

---

f. Dufay, "Mon chier amy" (ballade),
end of the second section

![Musical notation image]

---

g. End of a prelude from the Buxheim Organ Book

![Musical notation image]
Figure 9. Some formulas distinguishing $F^1$ and $F^2$

Only if $n = 12i + 7j$ (where $i > j$) can $1/n$ generate a system in $F^1$.

The following possibilities result when $i = 1, 2, 3$ or $4$ and $j = 0, 1$ or $2$:

Values for $i$: 1 2 3 4

Values for $j$: 0 12 24 (36) (48) 1 31 43 55 2 50 (62)

(The numbers in parentheses are multiples of smaller results in the same table and represent equal-division systems which have *not* played a very substantial role in the history of music theory.)

Only if $n = 12i + 7j$ (where $i > j$) can $1/n$ generate a system in $F^2$.

The following possibilities result when $i = 1$ or $4$ and $j = 0$ or $1$:

Values for $i$: 1 4

Values for $j$: 0 12 (48) 1 53
Figure 10. The V and III relations in Newton's system

\[
\begin{align*}
(G) & \quad \text{--} \quad (D) & \quad \text{--} \quad (A) & \quad \text{--} \quad (E) & \quad \text{--} \quad (B) & \quad \text{--} \quad (F\#) & \quad \text{--} \quad (C\#) \\
52 & \quad 30 & \quad 8 & \quad 39 & \quad 17 & \quad 48 & \quad 26
\end{align*}
\]

/ \quad / \quad / \quad / \quad / \quad / \quad /

\[
\begin{align*}
(Eb) & \quad \text{--} \quad (Bb) & \quad \text{--} \quad (F) & \quad \text{--} \quad (C) & \quad \text{--} \quad (G) & \quad \text{--} \quad (D) & \quad \text{--} \quad (A) & \quad \text{--} \quad (E) \\
35 & \quad 13 & \quad 44 & \quad 22 & \quad 53 & \quad 31 & \quad 9 & \quad 40
\end{align*}
\]

GIF 8. Some 18th-century irregular temperaments
GIF 9. Louis Couperin, Pavane in F#-minor, first section

GIF 10. The “margin of equivalence” between two physically different systems with the same number of pitch classes (in this case, four)

System A:

quasi-system:

System B:

margin of equivalence