Music Theory Online

The Online Journal of the

Society for Music Theory

Volume 9, Number 4, October 2003

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A Tetrahedral Graph of Tetrachordal Voice-Leading Space[1]

For Robert Morris

KEYWORDS: voice leading, tetrachord, simplex, atonality, P relation, transpositional combination

ABSTRACT: A tetrahedral graph models voice leading among the 29 T/I-type tetrachord classes. Transpositional combination plays a crucial role in the structure of the tetrahedron. A dipyramid, fusing two tetrahedra, models similar relations among the 43 T-type |4|-classes. The two graphs generalize to n-dimensional simplexes for relations among |*n* + 1|-classes, in modulo 12 and in other universes of even cardinality. A peculiarity is that the symmetrical graphs are a bit too large to hold the asymmetrical collection of abstract objects that they are designed to contain. A handful of set-class duplications serve as bubble wrap; the article devotes considerable attention to investigating their status.

Submission received July 2003

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[1] A version of this paper was presented at a symposium on Music and Mathematics organized by Robert Peck and Judith Baxter, as part of the southeast regional meeting of the American Mathematical Society in Baton Rouge, March 2003. Joseph Straus's unpublished graph prompted the questions that led to this research, and he generously permitted me to reproduce his graph concurrently with the publication under his name. Ian Quinn's unpublished version of Straus's graph suggested a central insight, and conversations with Ian have helped sharpen my understanding of many aspects of this presentation. Jack Douthett made a number of helpful suggestions and corrections to the proofs. A question from Panayotis Mavromatis stimulated the exploration undertaken in paragraphs [31] through [33]. Richard Plotkin created the graphics for this article. I also thank Clifton Callender, Robert Morris, John Roeder, Steve Soderberg, Zal Usiskin, and Jonathan Wild for suggestions and information. Julian Hook gave the entire manuscript a thorough reading and made many valuable suggestions.

[1] In recently published work, Joseph Straus presents a graph of close voice-leading relations among the twenty-nine T/I tetrachord classes (Straus 2003, 339; see Figure 1). Pairs of tetrachord classes (henceforth "|4|-classes") are adjacent in Figure 1 if a member set of one can be transformed into a member set of the other through a single semitonal displacement. The graph is not planar: it cannot be represented in two dimensions without intersecting edges. The current article introduces a tetrahedron that extends Straus's graph into three dimensions, and reveals certain symmetries latent in it. The tetrahedral model has several advantages: it groups |4|-classes in ways that interact with earlier and developing work; it calls attention to several Cinderella |4|-classes, whose charms have been overlooked; and it generalizes in several significant directions. Yet the model also raises a peculiar problem: it contains thirty-five |4|-classes, including six pairs of |4|-classes with identical prime forms. The ontological and phenomenal status of these apparent redundancies is still unclear.

[2] The paper begins with an informal tour of the tetrahedron, together with some preliminary observations, as a prelude to a more systematic presentation. Because a two-dimensional projection of a tetrahedron is difficult to "read," the model is introduced through a series of component sections, which in combination will build up an intuition of the whole. Readers who have difficulty imagining the entire Gestalt may find it useful to realize it "in real place" with Styrofoam balls and toothpicks.

Presentation of the Tetrahedron, and Initial Observations

[3] A tetrahedron is a polyhedron with four vertices, six edges, and four triangular faces. The vertices are labeled using the first four letters of the alphabet, and edges and faces are labeled by naming the pairs and triplets of vertices, respectively, that they connect. In the tetrahedron at hand, each edge contains five sites—three intermediate ones in addition to its two boundary vertices—and each face contains fifteen sites: twelve at the triangular perimeter, and three forming an embedded triangle at the center.

[4] In principle, its symmetry dictates that a tetrahedron not be regarded from a particular canonical point of orientation. The ABC face of the tetrahedron at hand has special properties, however, which encourage us to view it as standing at the base of a pyramid, and we shall adopt this orientation throughout this paper. One useful way of sectioning the tetrahedron is into a succession of layered planes parallel to the ABC face. Each higher layer consists of a triangle of successively shorter edges. The second layer (counting from the bottom) contains ten sites, nine of which constitute the triangle's perimeter, the tenth of which has the distinction of being the only site in the entire tetrahedron not to inhabit one of the four faces. Moving up through the pyramid, the number of sites per layer diminishes through a series of triangular numbers, containing 6, 3, and 1 sites respectively. The entire tetrahedron consists of thirty-five sites (the sum of the first five triangular numbers), six more than the number of |4|-classes under T/I equivalence.

[5] The tetrahedron's four vertices, shown in Figure 2, contain all-combinatorial tetrachords. Vertex A is inhabited by the chromatic |4|-class, [0123]. Vertices B and C both host [0167], an instance of the redundancy referred to in paragraph [1]. In order to distinguish the sites and their contents, we will label the tetrachord at C as *0167*, and display it with a blue-colored sphere in

the figure. Vertex D, at the top of the pyramid, contains [0127].[2]

[6] Figure 3 fills the interiors of the six edges. In this and subsequent figures, yellow connecting edges indicate a direct relationship of a single semitonal displacement. Sites connected by blue edges are related, less directly, by double displacement. The contents of each edge can be characterized by an expression whose single integer variable *x* increments from 1 to 5:

- **Edge AB** contains chords of the form {0, *x*, *x* + 1, 2*x* + 1}. These include [0123], [0235], [0347], [0459], and [056e]. Only the first three of these correspond to the prime form, and thus appear as a label in Figure 3; the final two are converted to their prime forms, [0158] and [0167] respectively.
- **Edge BC** contains chords of the form {0, *x*, 6, 6 + *x*}. These include [0167], [0268], [0369], [046t], and [056e] Again the final two are converted to their prime forms, [0268] and [0167] respectively. The five sites of this edge form a palindrome, a circumstance whose consideration is deferred for the moment.
- **Edge AC** contains chords of the form {0, 1, 1 + *x*, 2 + *x*};
- **Edge AD** contains chords of the form {0, 1, 2, 2 + *x*};
- **Edges BD** and **CD** both contain chords of the form {0, 1, 1 + *x*, 7}.

[7] Each triangular face fills its center with three classes that occupy no edge. The ABC face is shown in Figure 4; since this face is at the base, we are viewing it from beneath. The center of the **ABC face** contains 0246, 0257, and 0358. Looking at the face as a whole, we note that all of its constituent |4|-classes are T_nI-invariant, but that the converse does not hold: [0127] and [0248] do not appear on the ABC face. There exists, however, a finer sieve that filters out these latter two chords, leaving those on the ABC face. To establish this sieve, we define two functions. The first one assigns a dyad to an interval class in the familiar way; the second one establishes the skip-length-pairs of normal-order tetrachords.

> **Definition 1. Interval class (IC) function.** Where *x* and *y* are pitch-classes, and subtraction is mod 12, IC(*x, y*) = min(*x* – *y, y* –*x*).
>
> **Definition 2. Skip-length pair (SLP) function.** For normal order tetrachord X = {*a, b, c, d*}, SLP(X) = (IC(*a, c*), IC(*b, d*)).[3]

SLP maps the |4|-classes onto the integer pairs (*m, n*) such that $2 \leq m \leq n \leq 6$, for which there are fifteen possible values. The integer pairs on the ABC face are those where $m = n$. Paragraph [21] shows that the |4|-classes with identical skip-length-pairs are exactly those that can be generated via transpositional combination.

[8] A unique property of the ABC face is that none of its |4|-classes are directly related to any of the others via semitonal displacement. Their proximity derives instead from their mutual displacement of a third |4|-class. It is this feature, which will come into clear view when we build the pyramid up in layers, that motivates the use of blue connectors in Figure 4, and the special status of the ABC face at the base of the tetrahedron.

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[2] In Figure 2 and all subsequent presentations of the tetrahedron, the angle at vertex C is a right angle. This is a representational choice without theoretical consequence; an equilateral tetrahedron would have served just as well.
[3] Four of a tetrachord's six intervals are steps; the remaining two are skips. In terms of Clough & Myerson 1985, the skip intervals are those whose diatonic length is 2.

[9] As the ABD and ACD faces are closely related, we will treat them together. The **ABD** face fills its center with classes [0236], [0237], and [0148] (Figure 5), the **ACD** face with classes [0146], [0136], and [0135] (Figure 6). The relationship between the faces becomes apparent through considering their skip-length-pair profiles. Each of the fifteen possible skip-length pairs appears on each face exactly once. Figure 7 constitutes an SLP map for the ABD face (cf. Fig. 5). Inverting this map about its central column provides the corresponding map for the ACD face (Fig. 6).

[10] The remaining face, **BCD**, fills its center with classes 0248, 0258, and *0258* (Figure 8). Taken as a whole, the |4|-classes on the BCD face are those whose skip-length pairs take the form (*n,* 6), i.e. those |4|-classes that contain at least one tritone as a skip interval. Of special interest is the central column, which contains those sets that dispose their elements symmetrically around a central tritone. The entire face is symmetric about this column: all six of the "redundancies" that expand the 29-class |4|-universe to the 35-site tetrahedron are contained right here.

[11] The "queen bee" at the interior of the tetrahedron is [0247]. Uniquely among all |4|-classes, [0247] semitonally displaces to the maximum of eight distinct |4|-classes.[4] (Eight is the maximum because there are four voices, each of which may be displaced in two directions.) Figure 9 presents [0247] and its eight incident vertices.

[12] Before proceeding to a more formal presentation, the reader is invited to take an animated tour of the tetrahedron via Figure 10, which can be grabbed and spun.

Minimal State Change, P relations, and Graphs

[13] This section temporarily places the tetrahedron to the side, in order to make explicit what it is that the figure is modeling. Having done so, we will reconstruct the tetrahedron from a different perspective.

[14] "Minimal semitonal displacement" is a manifestation of the more general phenomenon of **minimal state change**, which can be characterized as an ordered pair of non-identical states separated by the smallest possible distance within their conceptual space. Minimal change is pan-stylistically fundamental to melody, comes into play in scale construction with the advent of *musica ficta*, and acquires a permanent systematic role when key signatures become notational standards. Lerdahl 2001 notes (p. 74) that the phenomenon is quite general, with bases in both the natural and psychological worlds. Minimal state change occurs in a weak form when distinct cardinality-equivalent sets differ in a single element. A stronger version of the relation occurs when, in addition, the elements that distinguish the two sets are separated by a unit value.

[15] In music, strong minimal change is most frequently charted in the chromatic space of twelve pitch classes, where it is defined (for sets of equal cardinality) as displacement of a single pitch class by semitone. Two sets so related will be said to be in the P relation.[5]

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[4] It was Ian Quinn who called this feature of [0247] to my attention. "Queen bee" was suggested by Jack Douthett. To my knowledge, [0247] has not been selected out for special attention in the literature on atonal theory. Among its other special properties is that it is the only asymmetric tetrachord to contain neither semitones nor tritones, and the only asymmetric tetrachordal subset of the pentatonic ME set.

[5] The same relation is referred to as "maximal smoothness" in Cohn 1996, and as "Single Semitonal Displacement"

> **Definition 3. P relation for Set Pairs.** Pcsets X and Y are P-related (X P Y) if $|X| = |Y|$ and $X \Delta Y \in [01]$, where $X \Delta Y$ is the symmetric difference of X and Y.

Fifth-related diatonic collections are in the P relation; so too are parallel major and minor triads. Both of these examples involve members of a single T/I (Forte) set-class, a special case. The more general case involves sets that are not necessarily set-class equivalent. The C natural minor and C harmonic minor scales exhibit the P relation. Callender 1998 observes that the whole tone scale {C, D, E, F♯, G♯, A♯} is P-related to the "Mystic" hexachord {C, D, E, F♯, A, B♭}, which in turn is P-related to the octatonic subset {C, Db, E, F♯, A, Bb}. Similarly Douthett and Steinbach 1998 observe a chain of P relations in the following progression of common seventh-chord types: <{C, E, G, B} P {C, E, G, B♭} P {C, E♭, G, Bb} P {C, E♭, G♭, B♭} P{C♭, E♭, G♭, B♭}>.

[16] If two pitch-class sets are P-related, then the set classes that contain them are P-related.

> **Definition 4. P relations for Set-class pairs**. SC(X) P SC(Y) if there is some $X \in$ SC(X), $Y \in$ SC(Y) such that X P Y.

We assume for now the usual equivalence relation based on transposition and inversion (as in Forte 1973). This allows us to make the following claims about Forte classes, based on the four cases illustrated in paragraph [15] above:

- the diatonic set-class 7-35 is P-related to itself;
- the triadic set-class 3-11 is P-related to itself;
- the whole tone hexachord class 6-35 is P-related to the Mystic class 6-34, which in turn is P-related to the octatonic subset-class 6-49;
- The class of major sevenths, 4-20, is P-related to the class of dominant / ø7 chords, 4-27, which in turn is P-related to the class of minor sevenths, 4-26.

[17] Douthett 1993 suggests generalizing P relations in a direction that will be useful.

> **Definition 5. P^2 relations for sets.** For sets X and Z, X P^2 Z if there exists some set Y such that X P Y P Z.

All sets are trivially P^2-related to themselves. Examples of non-trivial P^2 relations include the C major and D major diatonic collections (both P-related to the G-major collection), and the D major and B-flat major triads (both P-related to the D minor triad). The relation generalizes to set classes as in the case of the P relation. Thus, the class of major sevenths is P^2-related to the class of minor sevenths via their mutual P relation to the class of dominant sevenths.[6]

[18] Mathematical relations can be modeled via simple graphs if they are both "symmetric—that is, if for every ordered pair (*a, b*) in the relation, the ordered pair (*b, a*) is also in the relation—and irreflexive—no ordered pairs of the form (*a, a*) appear in the relation" (Childs 1998b, p. 30). These conditions apply to P relations among sets. For set classes, P is symmetric, but not generally irreflexive, as triads and diatonic scales exemplify. P is, however, irreflexive for set classes of even cardinality. In order to understand this claim, we introduce a function that sums the elements of a pitch-class set.

(SSD) in Cohn 2000.

[6] For further discussion of P^2 relations, see Callender 1998 and Childs 1998a.

Definition 6. Sum Function. For $X = \{x_1, x_2, \ldots, x_j\}$, $\mathrm{SUM}(X) = \sum_{n=1}^{j} x_n$.

If *X* P *Y*, then SUM(*X*) – SUM(*Y*) = ±1, and hence *X* and *Y* have sums of opposite parity (i.e., one sum is even, the other odd). If *X* is of even cardinality, then all transpositions and inversions of *X* will have the same parity as *X*. Thus no set of even cardinality is P-related to another member of its set class, and so P is irreflexive for set classes of even cardinality.

Transpositional Combination and the ABC face

[19] Three of the tetrahedron's four faces constitute connected P graphs. As indicated in [8], however, there exist no P relations among the |4|-classes that inhabit the ABC face. The pitch-class sums introduced in [18] help us to understand this circumstance. We noted in paragraph [7] that *X* = {*a, b, c, d*} is on the ABC face if IC(*a, c*) = IC(*b, d*), (i.e., if SLP(*X*) is of the form (*m, m*)). It follows that *c - a* = *d - b* (mod 2) and so $SUM(X) \equiv 0$ (mod 2). As noted in [18], one member of a P-related pair must have an odd pc-sum. Hence there can be no P relations among members of the ABC face.

[20] Figure 4 is a graph of the many non-trivial P^2 relations among members of the ABC face. Every set-class on the ABC face is P^2-related to its neighbor above and below, left and right, and on both diagonals, up to a maximum of eight such relations.

[21] Each chord on the ABC face has a skip-length-pair of form (*n, n*), and hence its expression as a set takes the form *J* = {*x, y, x + n, y + n*}. We now show that the set of *J*-sets is exactly the set of |4|-classes TC that are generable via transpositional combination. Following Cohn 1986, a TC |4|-class is one whose member sets are related by transposition to *TC* = {*0, m, n, m + n*}. Each such class is generated under addition by *m'* * *n'*, where *m'* = {0, *m*} and *n'* = {0, *n*}. The two skip intervals of *TC* are *n* – 0 = *n* and (*m* + *n*) – *m* = *n*, hence SLP(*TC*) = (*n, n*) = SLP(*J*). Conversely, *J* = {*x, y, x + n, y + n*} is transpositionally related to *TC* = {0, *m, n, m + n*}: define *m* = *y* – *x*; then *y* = *x* + *m*, and so *J* = {*x, x + m, x + n, (x + m) + n*}. Subtracting *x* from each element of *J* yields {0, *m, n, m + n*} = *TC* = $T_{-x}(J)$.

[22] Figure 11 presents the results for *m* * *n* under transpositional combination. The main diagonal, coded in blue, presents results for *m* = *n*; these are not tetrachords. The remaining thirty results are duplicated across that diagonal, in red and green respectively. These triangles of fifteen are rotationally or inversionally equivalent to Figure 4. Only thirteen distinct |4|-classes are represented. |4|-class [0167] appears in the table twice, as a product of 1 * 6 and also of 5 * 6. |4|-class [0268] also appears twice, as a product of 2 * 6 and of 4 * 6.[7] Although it is tempting to prune out these redundancies, which are not present in the Straus graph, we will nonetheless retain them in order to enable the tetrahedral symmetry, deferring consideration of their status until the final section of this paper.

[7] For discussion, see Cohn 1986, 95-96; Cohn 1988, 28-29, and especially Cohn 1991.

Building the Tetrahedron in Layers

[23] In this section, we construct the tetrahedron by adding a series of layers parallel to the ABC face, whose components we will henceforth refer to as the TC |4|-chords. The ABC face contains ten P^2 sub-graphs that are **complete**, i.e. each pair of vertices is in the P^2 relation. Four of these sub-graphs are triangles along the hypotenuse of Figure 4; their vertices have the TC form $\{n * n + 1, n * n + 2, n + 1 * n + 2\}$, where $1 \leq n \leq 4$. The remaining six sub-graphs are squares with intersecting diagonals at their centers; their vertices have the TC form $\{n * m, n + 1 * m, n * m + 1, n + 1 * m + 1\}$, where $1 \leq n \leq 3 \leq m \leq 5$, $m - n \leq 2$. For each such sub-graph *TC*, there is a unique |4|-class *L* such that *tc* P *L* for all $tc \in TC$. We will refer to these ten *L* classes as **links**.

[24] Figure 12 shows the tetrahedral position of these ten link classes in relation to the TC classes. The lower level of Figure 12 is an inversion of Figure 4, now viewed from above, and lacking the blue connectors that indicated P^2 relations on the earlier figure. At the second level, those link classes that are P-related to vertices along the AB edge are positioned directly above the center of a segment of that edge. Each of the remaining six link classes is positioned above the center of one of the local squares. The link |4| chords, like the TC chords, contain two redundancies: both [0258] and [0157] are represented twice, as indicated by the blue-colored spheres.

[25] Figure 12 contains two different types of connectors. The yellow connectors indicate P relations, which only exist between the two distinct levels. The light-blue ones indicate P^2 relations among the link chords. (The P^2 connections among chords at the lower TC level are omitted here, for legibility.) The graph of P^2 relations among the link |4|-classes is isomorphic to a sub-graph of Figure 4 where the AB edge is eliminated, along with all of its incident edges.

[26] But the chords on the TC plane do not exhaust the |4|-classes that are P-related to all of the vertices of a complete *L* sub-graph. For every *TC* such that l P *TC*, there is also a distinct |4|-class *C* such that l P *C*, for all $l \in L$. These six *C* chords form the third layer of the tetrahedron, which is added in Figure 13. I shall refer to the constituents of this third layer as the **cloud** |4|-classes: each is positioned directly above a TC |4|-class, upon which it "casts a shadow." The six TC |4|-classes that are shadowed by the clouds are those that appear neither on the AC nor the BC edge; they are bounded by a triangle with vertices at [0235], [0257], and [0158]. Note that the [0147] tetrachord appears twice as a cloud, casting a shadow on both [0257] and [0158]. If two |4|-classes are in a cloud/shadow relation, then the P adjacencies of the shadow are a subset of the P adjacencies of its cloud. For example, the P-adjacency set of [0347] is {[0236], [0247], [0148]}, all chords on the second (link) level. The P-adjacency set of its [0237] cloud consists of the same three |4|-classes, plus an additional three whose location and identity we will now characterize.

[27] To generate the fourth layer, we proceed as with the third, cloud layer, which stands in the same relation to the link layer as the latter does to the TC layer. The cloud |4|-classes all have even pc-sums, and hence lack internal P relations; but the six clouds do participate in a rich P^2 orgy, whose graph is isomorphic to a sub-graph of Figure 4. The P^2 graph for clouds consists of three complete sub-graphs, including two triangles and a square. For each of the three complete *C* sub-graphs, there is a link class *L* such that *c* P *L* for all $c \in C$. In addition, there is a distinct |4|-class *CL* such that *c* P *CL* for all $c \in C$. These three *CL* chords form the fourth layer of the tetrahedron, which is added in Figure 14. I shall refer to the constituents of this fourth layer as

the **cloud-link** |4|-classes. (The label is doubly determined: each of these three |4|-classes both links a cloud and clouds a link.) Each of the three cloud-links is positioned directly above a link |4|-class whose P-adjacencies it shares. The three link |4|-classes that are shadowed by the cloud-links are [0236], [0247], and [0148]. Note that the [0137] tetrachord appears twice on the cloud-link level, casting a shadow on both [0247] and [0148].

[28] The three cloud links on the fourth level constitute a complete P^2 graph, and each is P-related to [0237] on the third level. The same three cloud links constitute the complete set of P relations for the one |4|-class as yet absent from the tetrahedron: [0127]. This final class alone constitutes the **crowning** layer. Figure 15 completes the tetrahedron by adding this final class, suitably decked in red. Like the [0247] "queen bee" discussed in paragraph [12], the special status of the [0127] crown has been somewhat overlooked in the atonal pitch-class literature. It is one of the all-combinatorial tetrachords, indeed the only one not generable by transpositional combination (Cohn 1986, p. 122), and it is one of two T_nI-invariant |4|-classes that includes its only inversional axes.

[29] Each of the five layers of the tetrahedron has a distinct skip-interval profile. We have already noted that the TC classes that inhabit the lowest layer are those with skip-interval pairs of the form (n, n). The chords at the other layers have distinct skip intervals; the distance between them is constant within a given layer, and increases as we ascend through the layers. The ten chords of the link layer have skip-interval pairs of form $(n, n + 1)$; the six chords of the cloud layer have SLP-form $(n, n + 2)$; the SLP form of those three at the cloud-link layer is $(n, n + 3)$; and the crown partitions into skip intervals of the form $(n, n + 4)$.

Extensions and Generalizations

[30] The tetrahedral model has the potential to generalize in a number of directions, some of which may have intrinsic value for the understanding they shed on musical materials and relations that we care about, others of whose value may only be secondary, helping us to understand a larger and more abstract picture that bounces light back onto the specific structure already at hand. Four general directions look promising for further exploration, one by one and in combination.

[31] First, what does the |4|-class universe look like under parsimonious voice leading if we consider other possible construals of class-equivalence, along the lines of the set groups discussed in Morris 1982? Consider the geometry of the forty-three |4|-classes under transposition alone, taking as a basis the tetrahedron that has been our object of study so far. In order to provide a domicile for the fourteen additional classes, we build a second tetrahedron downward from the ABC face (which contains thirteen of the fifteen I-invariant |4|-classes). Each |4|-class in the original tetrahedron reflects into its inversion around the ABC face. The paired tetrahedra form a shape known as a triangular dipyramid.[8] Unlike the tetrahedron, each face of the dipyramid is a connected P graph. The ABC plane, which alone among the tetrahedral faces contained no P relations, is submerged in the dipyramid.

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[8] On triangular dipyramids, see the following internet site: http://mathworld.wolfram.com/TriangularDipyramid.html. Drag-clicking the figure spins it, with controllable direction and speed. My thanks to Ian Quinn for identifying this figure and introducing me to this site.

[32] The entire dipyramid contains fifty-five sites, indicating that there are twelve duplications, all of which are located on the BCD plane and its reflection, the BCE plane. Figure 16 pairs these two planes into a single projection, containing twenty-five sites in diamond shape. The entire diamond is symmetric about its central column. Note, however, that only ten duplications are accounted for by this symmetry. The remaining two occur in the central column, which is itself inversionally symmetric. The duplications are thus reflected in two different dimensions, indicating that the invariances of the plane cannot be described by any simple mapping.

[33] Figure 17 presents an improvement in this regard. It is derived by inverting the left half of the Figure 16 about its central row, so that the upper left and lower left quadrants exchange contents. Figure 17 is still a P-graph, as is its entire dipyramidal super-graph. But now the symmetry of the BCDE planar pair is simpler to describe: it maps into itself under 180º rotation, around the 0369 "pinwheel."

[34] A second general direction concerns the structure of P relations among members of other cardinality classes. Since complementary cardinalities have isomorphic P graphs, we need only concern ourselves with cardinalities ranging from 2 through 6.[9] The P graph of the |2|-classes is a line that connects [01] on one end to [06] on the other, and the P graph of T/I |3|-classes is a triangle (see Roeder 1987, p. 401). Lines, triangles, and tetrahedra are the simplex figures for the first three dimensions; by induction, we speculate that voice-leading parsimony among pentachord classes is modeled by a pentahedroid uniting five tetrahedra, and among hexachord classes by the five-dimensional simplex uniting six pentahedroids. Callender 2003 and Quinn 2003 have independently begun to explore this conjecture, which is easy to state but hard to intuit, harder yet to explore, yet again to demonstrate.
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[35] Until now we have assumed that a set {*a, b, c, d*} is a tetrachord only if it consists of four distinct pitch classes. A third direction for generalization lifts this restriction, integrating "classical" tetrachords (Quinn 2003) with four-element multi-sets, and enabling claims of the type {0, 2, 3, 5} P {0, 2, 2, 5}. A mapping of identical elements onto a single element would then allow exploration of P relations between sets that are not cardinality equivalent. Roeder 1987 presents a framework for such an exploration, in the context of trichordal multi-sets; Callender 2003 and Quinn 2003 have again taken the lead in treating the general case.

[36] A final direction for generalization involves P relations among set-classes in chromatic universes of diverse sizes. Such an exploration may seem irredeemably abstract for those for whom microtonality seems to hold limited aesthetic potential. Quite independently of these issues, expanding the universe of objects for study is a promising strategy for identifying general properties that can reflect specific insights onto the modulo-12 universe.[10] The following observations and conjectures about the number of |4|-classes in universes with an even number of elements, indeed, will provide ample confirmation of the tetrahedral model for modulo 12 despite its mysterious redundancies.

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[9] The tetrahedral graph is thus equally suitable as a map of P relations among octachord classes. On P relations and complementation, see Cohn 1996, p. 16.
[10] I address this issue in more depth in Cohn 1997, p. 23.

[37] Table 1 presents some data about the number of |4|-classes for universes of even cardinality of 8 and larger.[11] The variables in the table are defined as follows, where *N* is the set of natural numbers:

Definition 7. Triangular numbers Δ_n. For $n \in N$, $\Delta_n = 1 + 2 + ... + n$.

As is well known from number theory, the nth triangular number is equal to $\frac{n^2+n}{2}$.

Definition 8. Tetrahedral numbers J_n. For $n \in N$, $J_n = \Delta_1 + \Delta_2 + ... + \Delta_n$.

Again from number theory: the *n*th tetrahedral number is equal to $\frac{(n+1)^3-(n+1)}{6}$.

Definition 9. Dipyramidal numbers K_n. For $n \in N$, $K_n = J_n + J_{n-1}$.

We infer, from the formula for tetrahedral numbers, that the *n*th dipyramidal number is equal to $\frac{2n^3+3n^2+n}{6}$.

Definition 10. i_c is the number of |4|-classes, under T/I equivalence, in a *c*-element universe.

Definition 11. t_c is the number of |4|-classes, under T equivalence alone, in a *c*-element universe.

Definition 12. We will stipulate that *c*, the variable representing the cardinality of a given pitch-class universe, is equal to $2n + 2$.

Table 1

	size of chromatic universe ($c = 2n + 2$)	# of |4|-classes (TI type)	*n*th tetrahedral number	# of |4|-classes (T type)	*n*th dipyramidal number
n	c	i_c	J_n	t_c	K_n
3	8	8	10	10	14
4	10	16	20	22	30
5	12	29	35	43	55
6	14	47	56	73	91
7	16	72	84	116	140
8	18	104	120	172	204
9	20	145	165	245	285
10	22	195	220	335	385
11	24	256	286	446	506
12	26	328	364		
13	28	413	455		
14	30	511	560		

[11] 8 is the smallest universe where tetrachords are not complements of chords of smaller cardinality.

[38] Three of the variables, *c, J,* and *K,* are definitionally related to *n*. The *i* and *t* variables, however, were simply hand-generated by "brute force" methods up through $c = 24$. These figures have since been subsequently confirmed, and supplemented for $c > 24$, by Jonathan Wild, who has derived them by other means, and further confirmed by the formula given in Gamer 1981. Theorems 1 and 2 assert the relation between the number of |4|-classes, for T/I- and T-equivalence respectively, and the size of the chromatic universe.[12]

Definition 13. Floor and Ceiling Function.

$\lfloor x \rfloor$ is the largest integer not greater than *x*. $\lceil x \rceil$ is the smallest integer not less than *x*.

Theorem 1. Number of T/I |4|-classes for a universe of *c* elements, *c* even:

$$i_c = \left\lfloor \frac{c^3 - 3c^2 + 8c}{48} \right\rfloor = \begin{cases} \dfrac{c^3 - 3c^2 + 8c}{48} & \text{if } c \equiv 0 \pmod 4 \\ \dfrac{c^3 - 3c^2 + 8c - 12}{48} & \text{if } c \equiv 2 \pmod 4 \end{cases}$$

Theorem 2. Number of T |4|-classes for a universe of *c* elements, *c* even:

$$t_c = \left\lfloor \frac{c^3 - 6c^2 + 14c}{24} \right\rfloor = \begin{cases} \dfrac{c^3 - 6c^2 + 14c}{24} & \text{if } c \equiv 0 \pmod 4 \\ \dfrac{c^3 - 6c^2 + 14c - 12}{24} & \text{if } c \equiv 2 \pmod 4 \end{cases}$$

[39] We are now in a position to assert three theorems concerning the relationship between the geometric figures that are the focus of this paper and the shape of |4|-class space under voice-leading parsimony, for chromatic universes with an even number of pitch classes. Proofs of the three theorems are provided as appendices. Theorem 3 expresses the number of T/TI |4|-classes in terms of a tetrahedral number:

Theorem 3. Number of T/I |4|-classes for universe $c = 2n + 2$.

$$J_n - i_c = \left\lceil \frac{n^2 - 1}{4} \right\rceil$$

Theorem 4 expresses the number of T |4|-classes in terms of a dipyramidal number:

Theorem 4. Number of T |4|-classes for universe $c = 2n + 2$.

$$K_n - t_c = \left\lceil \frac{n^2 - 1}{2} \right\rceil$$

Theorem 5 compares the number of T and T/I |4|-classes for even cardinality *c*:

Theorem 5. $2i_c - t_c = \Delta_n$.

[12] Theorem 1 simplifies, for the cases at hand, the formulas given in Gamer 1981 and Reimer 1985.

[40] These theorems demonstrate that the tetrahedral model of the modulo-12 T/I |4|-class universe, together with its extension into the dipyramidal model for the modulo-12 T |4|-class universe, generalizes to the structure of tetrachords in all chromatic universes with an even number of elements. Preliminary exploration of $c = 8$, 10, and 14 suggests that a number of properties of the mod-12 tetrahedron/dipyramidal model have quite general scope. These properties include the following, where again $n = c/2 - 1$:

1. T/I-equivalent |4|-classes are modeled by a tetrahedron whose edges have n points.
2. The four vertices are inhabited by A = [0,1,2, 3], B = C = $\{0, 1, n + 1, n + 2\}$, D = $[0, 1, 2, n + 2]$.
3. For $1 \leq x \leq n$,
   Edge AB contains chords of the form $\{0, x, x + 1, 2x + 1\}$;
   Edge BC contains chords of the form $\{0, x, n + 1, n + 1 + x\}$.
   Edge AC contains chords of the form $\{0, 1, 1 + x, 2 + x\}$;
   Edge AD contains chords of the form $\{0, 1, 2, 2 + x\}$;
   Edges BD and **CD** both contain chords of the form $\{0, 1, 1 + x, n + 2\}$.
4. The |4|-classes on the ABC face are those that have identical skip-length pairs, which are equivalent to those generable by transpositional combination of dyads $x * y$, for $1 \leq x < y \leq n + 1$.
5. The |4|-classes on the BCD face are those whose skip-length pairs are of the form $(x, n + 1)$, for $2 \leq x \leq n + 1$.
6. The tetrahedron contains $\left\lceil \frac{n^2 - 1}{4} \right\rceil$ duplicate classes, all contained on the BCD face, which is inversionally symmetric around its central column.
7. The ABD and ACD faces both present the complete aggregate of skip-length pairs (x, y), for $2 \leq x \leq y \leq n + 1$.
8. Building up from the ABC base, the tetrahedron consists of n layers, the qth layer (from the bottom) of which consists of the Δ_{n+1-q} |4|-classes of skip-length pairs $(p, p + q - 1)$, for $2 \leq p \leq n + 1$, $1 \leq q \leq n$.
9. T-equivalent |4|-classes are modeled by a triangular dipyramid whose edges have n points.
10. On the triangular dipyramid, inversion about the ABC plane maps |4|-classes into their pitch-class inversions.
11. The BCD and BCE faces map into each other under 180º rotation.
12. The dipyramid contains $\left\lceil \frac{n^2 - 1}{2} \right\rceil$ duplicate classes, all contained on the BCDE planar projection.

The Problem of the Duplications

[41] Our motivation for retaining the duplicate |4|-classes in the tetrahedral and dipyramidal model has been essentially formal. The duplications insure the model's symmetry (together with its historically attendant aesthetic values, such as comfort, awe, and beauty), its status as a three-dimensional simplex (opening the door to a certain species of generalization), and its legibility (preventing crossed edges and allowing a maximum number of sites to appear on the exterior). Ultimately, however, we must confront the ontological or phenomenal status of these duplications. Are the [0167] |4|-chords at the B and C vertices, and the other duplicate chords in their respective neighborhoods on the BCD face, identical objects experienced in identical ways? If so, how does this circumstance affect the structure and interpretation of the model? If not, what is the significant distinction between their respective structures and functions, and on what grounds can this distinction be articulated and experienced?

[42] Assuming that they are identical in all the ways that matter, we have a motivation for reflecting those identities in the geometry, so long as it remains legible. These identifications can be executed by creasing the BCD face of the tetrahedron at its central column, and folding the corresponding points together. The remaining three faces are elastically distorted, and the interior of the BCD plane, no longer a face, recedes into the interior of the figure, but otherwise this transformation seems feasible and satisfactorily usable. It is more difficult, however, to intuit a corresponding transformation of the dipyramidal model of the T-type |4|-classes, whose duplicates do not all map into each other by simple reflection. In both cases, it is also possible to retain the duplicates in representation, but identify them in imagination. Such identifications are not unfamiliar: we are accustomed to imagining them in the case of the torus, which characteristically "unwraps" into a planar geometry for use on the two-dimensional page, usually accompanied by a warning that the user must vigilantly bear in mind that the Euclidean distances are illusory (e.g. Roeder 1987, Hyer 1995).

[43] Assuming now that the chords in the B and C neighborhoods of the BCD face represent distinct states, on what grounds can the distinction be articulated? The prime forms that inhabit the tetrahedron (resp. dipyramid) are high up on the ladder of abstraction. The figure could also be inhabited by lower-level representatives which inhabit a larger universe of objects. If so, then actual distinctions between the entities are masked by their mutual reduction to a single prime-form. One consequence of this position is that the universe of objects at this lower level of abstraction is enlarged, so that the tetrahedron (resp. dipyramid) would constitute only one of a number of (presumably homomorphic) figures that combine to cover the entire space of those objects.

[44] Roeder 1987 presents a promising approach along these lines. His universe consists of ordered sets of intervallic adjacencies. In such a universe, <D, F♯> and <F♯, D> are distinct, representing intervallic adjacencies of <4> and <8> respectively, so that the six dyad classes under T-equivalence augment to eleven classes (excluding multisets) under dyadic order-adjacency. Similarly, the T-type |3|-class [014] has six order-adjacency representatives (because there are six order permutations of its elements) and the T/I type has twelve such (since each permutation has an independent retrograde). The two-dimensional space inhabited by trichordal order-adjacencies is thus much larger than the two-dimensional space occupied by trichordal set-classes, and each set-class is represented on that larger space by multiple distinct permutations. Roeder demonstrates

that the entire space partitions into a set of regions, each of which contains a single order-adjacency representative from each of the |3|-classes. Moreover, the regions are homomorphically related, in the sense that equivalent |3|-classes are represented at equivalent sites.

[45] Two characteristics of Roeder's space make it particularly promising for the problem at hand. First, although P relations are not of primary interest to Roeder, horizontal, vertical, or diagonal adjacency in his space is necessarily a P relation (provided that multisets are not involved); conversely, a P relation is necessarily an adjacency. Hence, his space easily converts to a P graph of trichordal order-adjacency series, and each region in his space converts to a P graph of trichordal set-classes. Second, redundancies arise at the boundaries of Roeder's regions. For example, <2,5> and <5,2>, both representing [027], arise at opposite boundaries of a single T-type trichordal region. Roeder stipulates that one of these chords be selected for the region at hand, while the other is assigned to the neighboring region.

[46] Here lies a potential solution to the problem of the duplications. By induction from the trichordal case, one might conjecture the following: 1) that an application of Roeder's method to tetrachords would yield a three-dimensional P-graph of tetrachordal order-adjacency space; 2) that the space would be sectioned into sub-regions, each of which is a complete P graph of |4|-classes; 3) that the sub-regions would be homomorphic, in the sense that each |4|-class would occupy an identical position in the geometry of the local sub-region; 4) that each T-type region would be a triangular dipyramid, and each TI-type region a tetrahedron; and 5) that duplicate classes arise at the boundaries, and are resolved through assignment to adjacent regions.

[47] In the event, only the first two of these conjectures hold. Roeder's sketch of tetrachordal order-adjacency space confirms that the space exists in three dimensions, that it is a P graph of tetrachordal order-adjacencies, that it can be sectioned into T- and TI-type |4|-chords such that each |4|-class is represented once per region, and that there are some duplications at the region boundaries. So far, so good. However, the equations that define the regions indicate that they do not have identical internal structures. The existence of an ordered tetrachord on a boundary of one of Roeder's region does not guarantee that its permutations will also appear on a boundary.[13] Moreover, Roeder's TI-type regions do not take the form of a tetrahedron.[14] The relation of Roeder's geometry to mine, however, warrants continued exploration. It is possible that my tetrahedra compound to form a cube related to Roeder's order-adjacency space, or that there exists a transformation that maps his regions into tetrahedra. It is also possible that neither of these circumstances is the case, but that Roeder's method of assigning duplications to neighboring structures in some space nonetheless holds the key to understanding the tetrahedral redundancies. Or perhaps, yet again, that the underlying geometry of tetrachordal space is not as symmetric as the representation put forth in this article suggests.

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[13] In order to exist on a boundary, an order-adjacency series must satisfy one of the following modulo-12 equations, given by Roeder on p. 397: $2x + y + z = 0$; $x + 2y + z = 0$; $x + y + 2z = 0$; $x + y + z = 0$; or $x = z$. {C♯, F♯, C, A♭}, whose adjacency series is <5, 6, 8>, satisfies $2x + y + z = 0$, and therefore stands at a boundary of a region; but {C♯, C, F♯, A♭}, series <11, 6, 2>, satisfies none of the equations and therefore is interior to a region.

[14] Roeder, private communication, 2003. According to Roeder's equations, both all-interval |4|-classes must occur at the interior of a section, whereas the tetrahedron has only a single site, containing [0247], at its interior.

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Appendices

Appendices 1 and 2 rely on the following properties of floor and ceiling functions, as given in Douthett 1999 (page 6):

(1) $\lfloor x + m \rfloor = \lfloor x \rfloor + m$, where m is an integer

(2) $\lceil -x \rceil = -\lfloor x \rfloor$

Appendix 1. Proof that $J_n - i_c = \left\lceil \frac{n^2-1}{4} \right\rceil$, where J_n is the nth tetrahedral number and i_c is the number of T/I - type | 4 | -classes in a universe of size $2n+2$.

$i_c = \left\lfloor \frac{c^3 - 3c^2 + 8c}{48} \right\rfloor$ by theorem 1;

$c = 2n + 2$ by definition 6;

$$i_c = \left\lfloor \frac{(2n+2)^3 - 3(2n+2)^2 + 8(2n+2)}{48} \right\rfloor$$

$$= \left\lfloor \frac{8n^3 + 12n^2 + 16n + 12}{48} \right\rfloor$$

$$= \left\lfloor \frac{2n^3 + 3n^2 + 4n + 3}{12} \right\rfloor;$$

$J_n = \frac{(n+1)^3 - (n+1)}{6}$ by definition 8;

$$= \frac{2(n+1)^3 - 2(n+1)}{12}$$

$$= \frac{2n^3 + 6n^2 + 4n}{12};$$

$$J_n - i_c = \frac{2n^3 + 6n^2 + 4n}{12} - \left\lfloor \frac{2n^3 + 3n^2 + 4n + 3}{12} \right\rfloor$$

$$= \frac{2n^3 + 6n^2 + 4n}{12} + \left\lceil -\left(\frac{2n^3 + 3n^2 + 4n + 3}{12} \right) \right\rceil$$

J_n is a tetrahedral number, hence an integer, and so

$$J_n - i_c = \left\lceil \frac{2n^3 + 6n^2 + 4n}{12} - \frac{2n^3 + 3n^2 + 4n + 3}{12} \right\rceil$$

$$= \left\lceil \frac{3n^2 - 3}{12} \right\rceil$$

$$= \left\lceil \frac{n^2 - 1}{4} \right\rceil.$$

Appendix 2. Proof that $\mathrm{K}_{\mathrm{n}} - \mathrm{t}_{\mathrm{c}} = \left\lceil \frac{n^2 - 1}{2} \right\rceil$, where K_n is the nth dipyramidal number and t_c is the number of T -type | 4 | -classes in a universe of size $2n + 2$.

$$t_c = \left\lfloor \frac{c^3 - 6c^2 + 14c}{24} \right\rfloor \text{ by theorem 2;}$$

$$c = 2n + 2 \text{ by definition 6;}$$

$$t_c = \left\lfloor \frac{(2n+2)^3 - 6(2n+2)^2 + 14(2n+2)}{24} \right\rfloor$$

$$= \left\lfloor \frac{8n^3 + 4n + 12}{24} \right\rfloor$$

$$= \left\lfloor \frac{2n^3 + n + 3}{6} \right\rfloor;$$

$$K_n = \frac{2n^3 + 3n^2 + n}{6} \text{ by definition 9;}$$

$$K_n - t_c = \frac{2n^3 + 3n^2 + n}{6} - \left\lfloor \frac{2n^3 + n + 3}{6} \right\rfloor$$

$$= \frac{2n^3 + 3n^2 + n}{6} + \left(\left\lceil -\frac{2n^3 + n + 3}{6} \right\rceil \right)$$

K_n is an integer, and so

$$K_n - t_c = \left\lceil \frac{2n^3 + 3n^2 + n}{6} - \frac{2n^3 + n + 3}{6} \right\rceil$$

$$= \left\lceil \frac{3n^2 - 3}{6} \right\rceil$$

$$= \left\lceil \frac{n^2 - 1}{2} \right\rceil.$$

Appendix 3. Proof that $2i_c - t_c = \Delta_n$.

Following definition 12, c is even, and hence is equivalent to either 0 or 2, modulo 4.

Taking first the case where $c \equiv 0 \pmod 4$:

From Theorem 1, $i_c = \dfrac{c^3 - 3c^2 + 8c}{48}$, and so $2i_c = \dfrac{c^3 - 3c^2 + 8c}{24}$;

From Theorem 2, $t_c = \dfrac{c^3 - 6c^2 + 14c}{24}$;

$$2i_c - t_c = \frac{c^3 - 3c^2 + 8c}{24} - \frac{c^3 - 6c^2 + 14c}{24}$$

$$= \frac{3c^2 - 6c}{24}.$$

Taking now the second case, where $c \equiv 2 \pmod 4$

From Theorem 1, $i_c = \dfrac{c^3 - 3c^2 + 8c - 12}{48}$, and so $2i_c = \dfrac{c^3 - 3c^2 + 8c - 12}{24}$;

From Theorem 2, $t_c = \dfrac{c^3 - 6c^2 + 14c - 12}{24}$

$$2i_c - t_c = \frac{c^3 - 3c^2 + 8c - 12}{24} - \frac{c^3 - 6c^2 + 14c - 12}{24}$$

$$= \frac{3c^2 - 6c}{24}$$

Hence the two cases are united for the remainder of the proof.

Following definition 12, we substitute $2n + 2$ for c:

$$2i_c - t_c = \frac{3(2n+2)^2 - 6(2n+2)}{24}$$

$$= \frac{12n^2 + 24n + 12 - (12n + 12)}{24}$$

$$= \frac{12n^2 + 12n}{24}$$

$$= \frac{n^2 + n}{2}$$, which is Δ_n according to definition 7

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prepared by
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Updated 22 October 2003