

MTO 13.4 Examples: Hook, Why are There Twenty-Nine Tetrachords?

(Note: audio, video, and other interactive examples are only available online)

<http://www.mtosmt.org/issues/mto.07.13.4/mto.07.13.4.hook.php>

Table 1. Enumeration of mod-12 pitch-class sets and T_n/T_nI set classes

<i>Cardinality</i>	0	1	2	3	4	5	6	7	8	9	10	11	12	<i>Total</i>
<i>No. of pc-sets</i>	1	12	66	220	495	792	924	792	495	220	66	12	1	4096
<i>No. of set classes</i>	1	1	6	12	29	38	50	38	29	12	6	1	1	224

Figure 1. Pascal's triangle

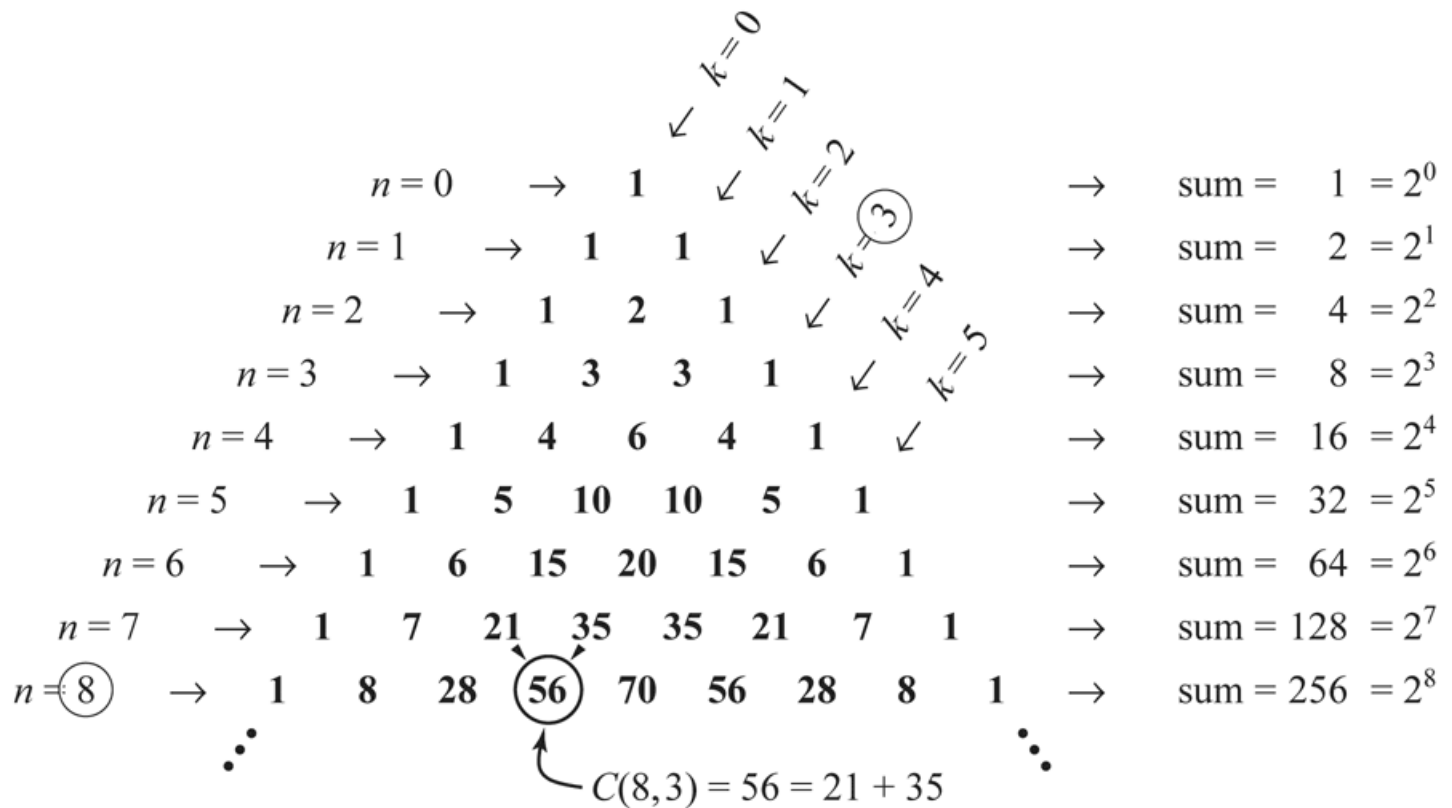


Figure 2. The binomial theorem

$$\begin{aligned}(x + y)^0 &= \mathbf{1} \\(x + y)^1 &= \mathbf{1}x + \mathbf{1}y \\(x + y)^2 &= \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2 \\(x + y)^3 &= \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3 \\(x + y)^4 &= \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4 \\&\vdots \\(x + y)^n &= C(n, 0)x^n y^0 + C(n, 1)x^{n-1}y^1 + C(n, 2)x^{n-2}y^2 \\&\quad + \dots + C(n, k)x^{n-k}y^k + \dots \\&\quad + C(n, n-2)x^2y^{n-2} + C(n, n-1)x^1y^{n-1} + C(n, n)x^0y^n\end{aligned}$$

Figure 3. Four configurations of candles on a birthday cake or beads in a necklace

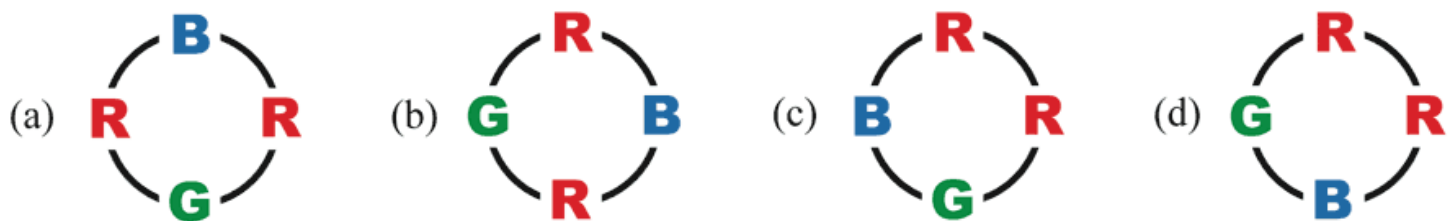


Figure 4. The transformations in the dihedral group D_4

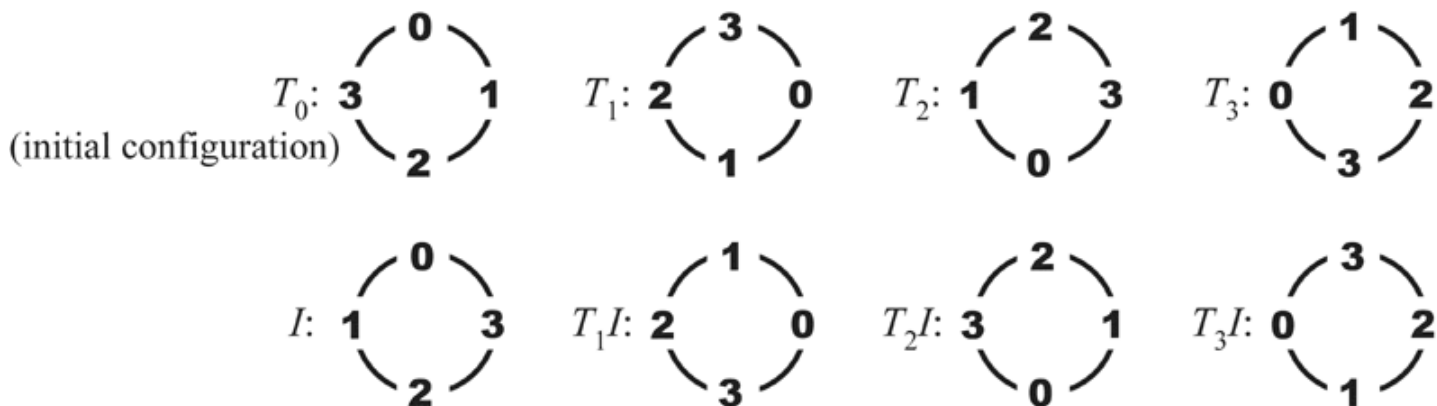


Table 2. Cycle structure of the permutations in the dihedral group D_4 (3-color necklace problem)

<i>Transformation</i>	<i>Action</i>	<i>Cycle representation</i>	<i>No. of cycles</i>	<i>Fixed configs.</i>	<i>Cycle type</i>	<i>Cycle index</i>
T_0	$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3$	$(0)(1)(2)(3)$	4	$3^4 = 81$	1^4	t_1^4
T_1	$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 0$	$(0\ 1\ 2\ 3)$	1	$3^1 = 3$	4^1	t_4^1
T_2	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1$	$(0\ 2)(1\ 3)$	2	$3^2 = 9$	2^2	t_2^2
T_3	$0 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$	$(0\ 3\ 2\ 1)$	1	$3^1 = 3$	4^1	t_4^1
I	$0 \rightarrow 0, 1 \rightarrow 3 \rightarrow 1, 2 \rightarrow 2$	$(0)(1\ 3)(2)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_1 I$	$0 \rightarrow 1 \rightarrow 0, 2 \rightarrow 3 \rightarrow 2$	$(0\ 1)(2\ 3)$	2	$3^2 = 9$	2^2	t_2^2
$T_2 I$	$0 \rightarrow 2 \rightarrow 0, 1 \rightarrow 1, 3 \rightarrow 3$	$(0\ 2)(1)(3)$	3	$3^3 = 27$	$1^2 2^1$	$t_1^2 t_2^1$
$T_3 I$	$0 \rightarrow 3 \rightarrow 0, 1 \rightarrow 2 \rightarrow 1$	$(0\ 3)(1\ 2)$	2	$3^2 = 9$	2^2	t_2^2

Table 3. Cycle structure of the permutations in the dihedral group D_{12} (set-class problem)

Transformation	Cycle representation	No. of		Cycle type	Cycle index
		cycles	Fixed configs.		
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(A)(B)	12	$2^{12} = 4096$	1^{12}	t_1^{12}
T_1	(0 1 2 3 4 5 6 7 8 9 A B)	1	$2^1 = 2$	12^1	t_{12}^1
T_2	(0 2 4 6 8 A)(1 3 5 7 9 B)	2	$2^2 = 4$	6^2	t_6^2
T_3	(0 3 6 9)(1 4 7 A)(2 5 8 B)	3	$2^3 = 8$	4^3	t_4^3
T_4	(0 4 8)(1 5 9)(2 6 A)(3 7 B)	4	$2^4 = 16$	3^4	t_3^4
T_5	(0 5 A 3 8 1 6 B 4 9 2 7)	1	$2^1 = 2$	12^1	t_{12}^1
T_6	(0 6)(1 7)(2 8)(3 9)(4 A)(5 B)	6	$2^6 = 64$	2^6	t_2^6
T_7	(0 7 2 9 4 B 6 1 8 3 A 5)	1	$2^1 = 2$	12^1	t_{12}^1
T_8	(0 8 4)(1 9 5)(2 A 6)(3 B 7)	4	$2^4 = 16$	3^4	t_3^4
T_9	(0 9 6 3)(1 A 7 4)(2 B 8 5)	3	$2^3 = 8$	4^3	t_4^3
T_{10}	(0 A 8 6 4 2)(1 B 9 7 5 3)	2	$2^2 = 4$	6^2	t_6^2
T_{11}	(0 B A 9 8 7 6 5 4 3 2 1)	1	$2^1 = 2$	12^1	t_{12}^1
I	(0)(1 B)(2 A)(3 9)(4 8)(5 7)(6)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{1I}	(0 1)(2 B)(3 A)(4 9)(5 8)(6 7)	6	$2^6 = 64$	2^6	t_2^6
T_{2I}	(0 2)(1)(3 B)(4 A)(5 9)(6 8)(7)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{3I}	(0 3)(1 2)(4 B)(5 A)(6 9)(7 8)	6	$2^6 = 64$	2^6	t_2^6
T_{4I}	(0 4)(1 3)(2)(5 B)(6 A)(7 9)(8)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{5I}	(0 5)(1 4)(2 3)(6 B)(7 A)(8 9)	6	$2^6 = 64$	2^6	t_2^6
T_{6I}	(0 6)(1 5)(2 4)(3 7 B)(8 A)(9)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{7I}	(0 7)(1 6)(2 5)(3 4)(8 B)(9 A)	6	$2^6 = 64$	2^6	t_2^6
T_{8I}	(0 8)(1 7)(2 6)(3 5)(4)(9 B)(A)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{9I}	(0 9)(1 8)(2 7)(3 6)(4 5)(A B)	6	$2^6 = 64$	2^6	t_2^6
T_{10I}	(0 A)(1 9)(2 8)(3 7)(4 6)(5)(B)	7	$2^7 = 128$	$1^2 2^5$	$t_1^2 t_2^5$
T_{11I}	(0 B)(1 A)(2 9)(3 8)(4 7)(5 6)	6	$2^6 = 64$	2^6	t_2^6

Table 4. Values of $\zeta_N(k)$, the number of T_n classes of k -note chords in an N -note scale

	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	Total
$N=7$	1	1	3	5	5	3	1	1						20
8	1	1	4	7	10	7	4	1	1					36
9	1	1	4	10	14	14	10	4	1	1				60
10	1	1	5	12	22	26	22	12	5	1	1			108
11	1	1	5	15	30	42	42	30	15	5	1	1		188
12	1	1	6	19	43	66	80	66	43	19	6	1	1	352
13	1	1	6	22	55	99	132	132	99	55	22	6	1	632
14	1	1	7	26	73	143	217	246	217	143	73	26	7	1182
15	1	1	7	31	91	201	335	429	429	335	201	91	31	2192
16	1	1	8	35	116	273	504	715	810	715	504	273	116	4116
17	1	1	8	40	140	364	728	1144	1430	1430	1144	728	364	7712
18	1	1	9	46	172	476	1038	1768	2438	2704	2438	1768	1038	14602
19	1	1	9	51	204	612	1428	2652	3978	4862	4862	3978	2652	27596
20	1	1	10	57	245	776	1944	3876	6310	8398	9252	8398	6310	52488
21	1	1	10	64	285	969	2586	5538	9690	14000	16796	16796	14000	99880
22	1	1	11	70	335	1197	3399	7752	14550	22610	29414	32066	29414	190746
23	1	1	11	77	385	1463	4389	10659	21318	35530	49742	58786	58786	364724
24	1	1	12	85	446	1771	5620	14421	30667	54484	81752	104006	112720	699252

Table 5. Values of $\xi_N(k)$, the number of T_n/T_nI classes of k -note chords in an N -note scale

	$k=0$	1	2	3	4	5	6	7	8	9	10	11	12	Total
$N=7$	1	1	3	4	4	3	1	1						18
8	1	1	4	5	8	5	4	1	1					30
9	1	1	4	7	10	10	7	4	1	1				46
10	1	1	5	8	16	16	16	8	5	1	1			78
11	1	1	5	10	20	26	26	20	10	5	1	1		126
12	1	1	6	12	29	38	50	38	29	12	6	1	1	224
13	1	1	6	14	35	57	76	76	57	35	14	6	1	380
14	1	1	7	16	47	79	126	133	126	79	47	16	7	687
15	1	1	7	19	56	111	185	232	232	185	111	56	19	1224
16	1	1	8	21	72	147	280	375	440	375	280	147	72	2250
17	1	1	8	24	84	196	392	600	750	750	600	392	196	4112
18	1	1	9	27	104	252	561	912	1282	1387	1282	912	561	7685
19	1	1	9	30	120	324	756	1368	2052	2494	2494	2052	1368	14310
20	1	1	10	33	145	406	1032	1980	3260	4262	4752	4262	3260	27012
21	1	1	10	37	165	507	1353	2829	4950	7105	8524	8524	7105	50964
22	1	1	11	40	195	621	1782	3936	7440	11410	14938	16159	14938	96909
23	1	1	11	44	220	759	2277	5412	10824	17930	25102	29624	29624	184410
24	1	1	12	48	256	913	2920	7293	15581	27407	41272	52234	56822	352698