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MTO 0.3 Examples: Lindley and Turner-Smith, Mathematical Models of Scales

(Note: audio, video, and other interactive examples are only available online)
http://www.mtosmt.org/issues/mto.93.0.3/mto.93.0.3.lindley_turner-smith.php

Figure 1. A sequence of constructions abstracted from the pitch continuum

```
a) a positive-number line for pitch frequencies

b) a number line for logarithms to base 2 of the frequencies

c) equivalence classes of points mod 1 on this number-line (flogs)

d) pitch-class relations between these equivalence classes (also flogs)

e) two kinds of generators for groups of pitch-class relations:
   equal-division (1/n converted to a flog) or harmonic (see below)

f) the pairs, (set, Abelian group), which ensue from these generators

g) equivalence-class neighborhoods around every point-mod-1 in this set

h) "ideal systems", whereby every system comprises a pair:
   a finite set of non-overlapping neighborhoods, operated upon by
   a subset of one of our groups (that is, by a "halfgroup")

i) unlimited scales (repeating in every octave indefinitely)

j) scales with a highest and lowest note
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Figure 2. Three pitches in one pitch class

```
                    (octave)                (octave)
    pitches:   ----+-----------------------+-----------------------+---

                   r                      2r                      4r
 frequencies:  ----+-----------------------+-----------------------+---

                   r                     r + 1                   r + 2
   our model:  ----+-----------------------+-----------------------+---
```

GIF 1. How to reckon in flogs rather than logs to base 2

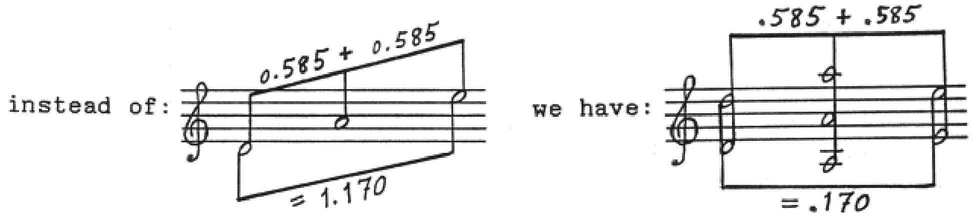

Figure 3. Three notes in one pitch class

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                 octave                                  octave
---(-+-)--------------------(-+-)--------------------(-+-)---
   note                     note                     note
```

GIF 2. Some types of algebraic structure

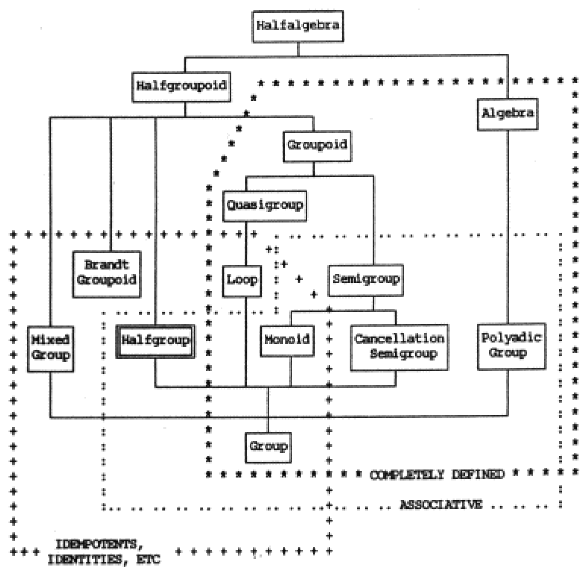

Figure 4. Our notation for the generating harmonic pitch-class relations

```
  "I"  = flog 2 (= 0, thus the identity element)
  "V"  = flog 3 plus or minus a much smaller flog (t^V^)
"III" = flog 5 plus or minus a much smaller flog (t^III^)
"VII" = flog 7 plus or minus a much smaller flog (t^VII^)
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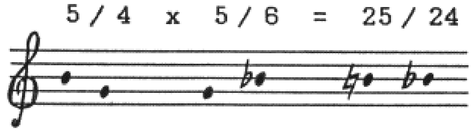
Figure 5. A rough classification of orders of intervallic magnitude

```
     (ca. 10 octaves  -  range of hearing)
       (ca. 1 octave  -  difference between men's and women's voices)
  10ths of an octave  -  melodic steps and leaps
 100ths of an octave  -  out-of-tune-ness (Such a t would mar a consonance.)
1000ths of an octave  -  tempering
     1/10 000-octave  -  musically insignificant
```

GIF 3. Reckoning the size of a chromatic semitone

5 / 4 x 5 / 6 = 25 / 24



log (25/24) / log 2 = o.o6

Figure 6. A table showing how to find the most feasible equations nV = III

```
                                                 T^n^ =
n      m(flog 3)    flog 5  =  s^n^     m^n^    s^n^/(m^n^ + 1)

1       -flog 3   - flog 5   = .o931      1         .o466
2      3 flog 3   + flog 5   = .o768      3         .o192
3      4 flog 3   - flog 5   = .o179      4         .oo36             -->  4V = III
4      8 flog 3   + flog 5   = .oo16      8         .ooo2             --> -8V = III
5     45 flog 3   - flog 5   = .oo14     45         .oooo3
```

Figure 7. Branches in our system-tree

a) Among systems:
harmonic
equal-division

b) Among harmonic systems:
1-dimensional (the generators are {I, V})
2-dimensional (the generators are {I, V, III})
3-dimensional (the generators are {I, V, III, VII})

c) Also among harmonic systems:
coherent (comprising one chain of V's)
not coherent

d) Among coherent systems:
untempered (all t = 0)
tempered

e) Among temperaments:
regular (all t^V^ equal, all t^III^ equal, etc.)
semi-regular (all t^V^ equal, but not all t^III^ or t^VII^)
irregular (t^V^ and hence t^III^ & t^VII^ varying)

f) Among regular, two- (and sometimes three-) dimensional temperaments:
MT (wherein 4V = III)
QP (wherein -8V = III)

g) Conjunctions of MT and QP:
ET^1^ (12V = I; there is no III and no VII)
ET^2^ (12V = I; 4V = III = -8V; there is no VII)
ET^3^ (12V = I; 4V = III = -8V; -2V = VII = 10V)

h) Apart from ET, some musically good possibilities for MT or for QP:
meantone temperaments
quasi-Pythagorean temperaments

i) Physically equivalent equal-division systems,
for instance, the mere division of the octave into 12 equal parts
(a system which is physically equivalent to an ET but has no diatonic
or chromatic semitones)

j) Certain families of such equal-division systems:
F^1^ (characterized by equivalence to MT systems)
F^2^ (characterized by equivalence to QP systems)

k) A kind of temperament, CT, which includes ET and some irregular temperaments
that approximate to an ET within a certain "margin of equivalence"
and meet certain other specifications

l) Some historically important kinds of CT:
JSB
temperament ordinaire (with an accent over the second "e")

m) A "semi-regular" temperament that is physically equivalent to a
quasi-Pythagorean temperament but has two different values for III

Figure 8. Diagram of a meantone system with 14 pitch classes

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("--" between two note-names means there is a V relation between the
 two pitch classes; "/" means there is a III relation between them.
 Imagine the "--"s spiraling up on a cylinder from Ab to D#)


            C#  --  G#  --  D#

          /       /       /

        A   --  E   --  B   --  F#  -- (C#)

      /       /       /       /

    F   --  C   --  G   --  D   --  (A)

          /       /       /

        Ab  --  Eb  --  Bb  --  (F)
```
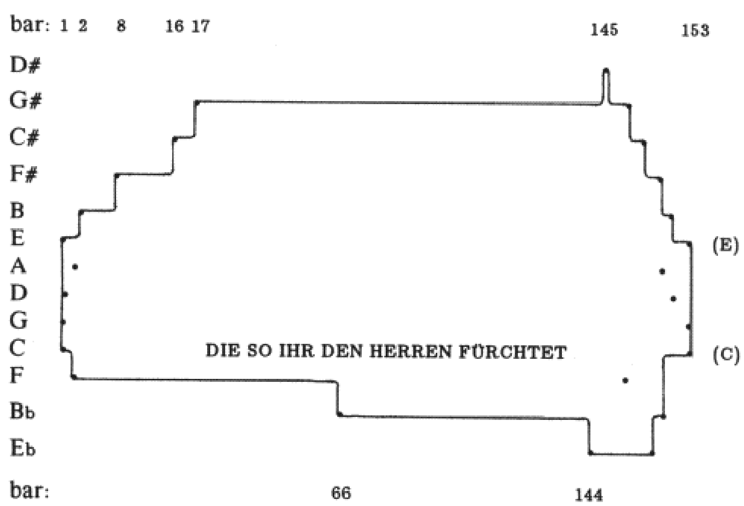
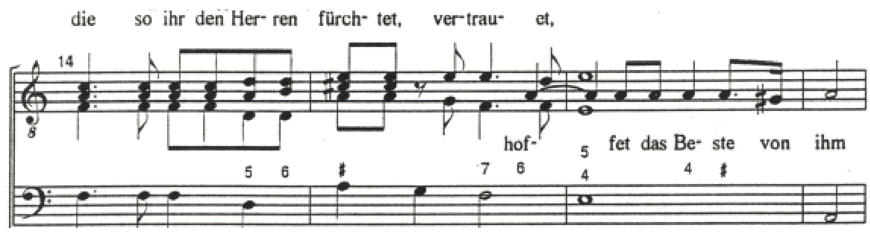

GIF 4. Schütz, "Die so ihr den Herren fürchtet" (SWV 164)

(a) A diagram showing when the first and last use of each pitch class occurs



(b-d) Three excerpts showing some of these uses b.

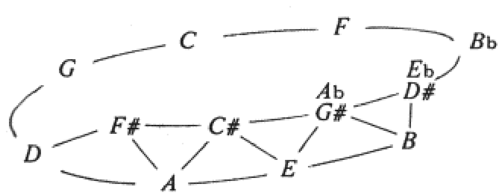
c.



d.

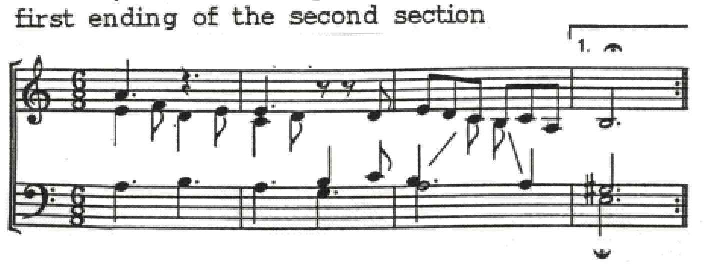
GIF 5. A 15th-century quasi-Pythagorean system and some evidence of its use: music in which triads with an explicit sharp are especially salient

a.



b.

Landini, "O fanciulla giulìa" (ballata), first ending of the second section



c. Matteo da Perugia, "A qui fortune" (rondeau), end of the first section

d.

Matteo da Perugia, "Le grant desir" (ballade), end of the first section

e.

End of an organ verse from the Faenza Codex

f. Dufay, "Mon chier amy" (ballade), end of the second section

g.

End of a prelude from the Buxheim Organ Book

GIF 6. Lewin's analysis of a passage from *Parsifal*

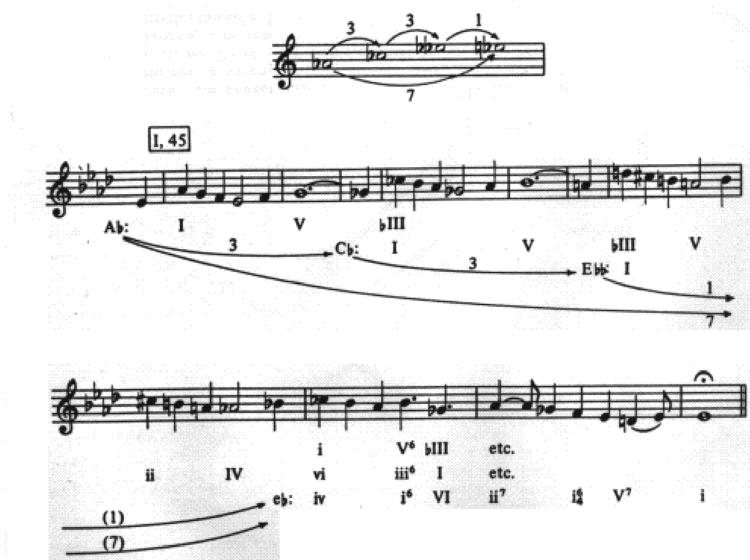


Figure 9. Some formulas distinguishing F^1 and F^2

```
Only if n = 12i + 7j (where i>j) can 1/n generate a system in F^1^.

The following possibilities result when i = 1, 2, 3 or 4
                                     and j = 0, 1 or 2:

                    Values for i:  1    2    3    4

              Values for j: 0     12   24  (36) (48)
                            1          31   43   55
                            2               50  (62)

(The numbers in parentheses are multiples of smaller results in the
 same table and represent equal-division systems which have *not*
 played a very substantial role in the history of music theory.)


Only if n = 12i + 7j (where i>j) can 1/n generate a system in F^2^.

The following possibilities result when i = 1 or 4
                                     and j = 0 or 1:

                             Values for i:  1   4

                       Values for j: 0     12  (48)
                                     1          53
```

GIF 7. A diagram made by Isaac Newton

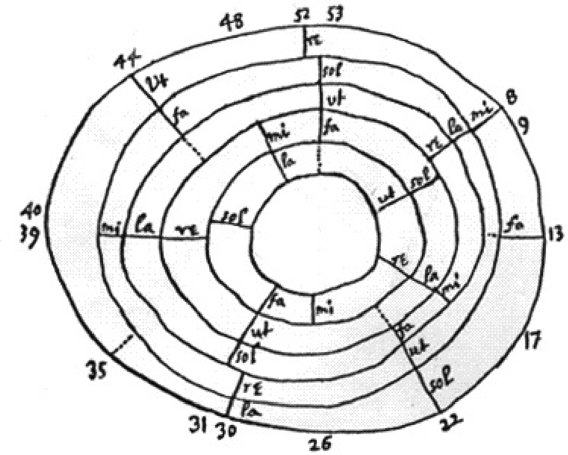
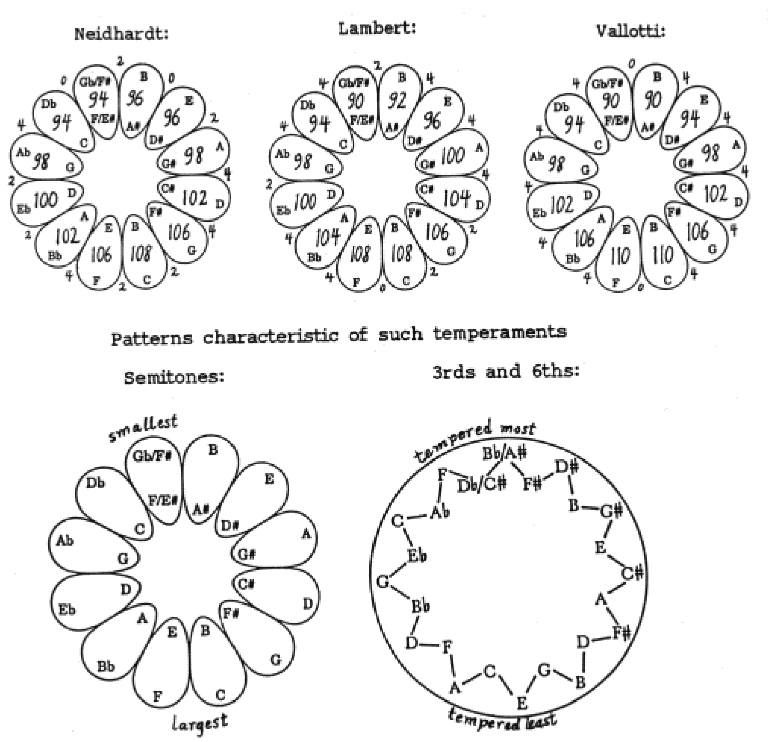


Figure 10. The V and III relations in Newton's system

```
      (G)    --    (D)   --   (A) --   (E)   --   (B) -- (F#)   -- (C#)
      52           30          8       39         17      48         26

    /          /          /          /         /         /          /

(Eb)  --    (Bb)  --   (F)   --   (C)   -- (G)   -- (D)   --   (A)   --   (E)
 35          13         44         22        53        31        9          40
```

GIF 8. Some 18th-century irregular temperaments



GIF 9. Louis Couperin, Pavane in F#-minor, first section

GIF 10. The "margin of equivalence" between two physically different systems with the same number of pitch classes (in this case, four)