

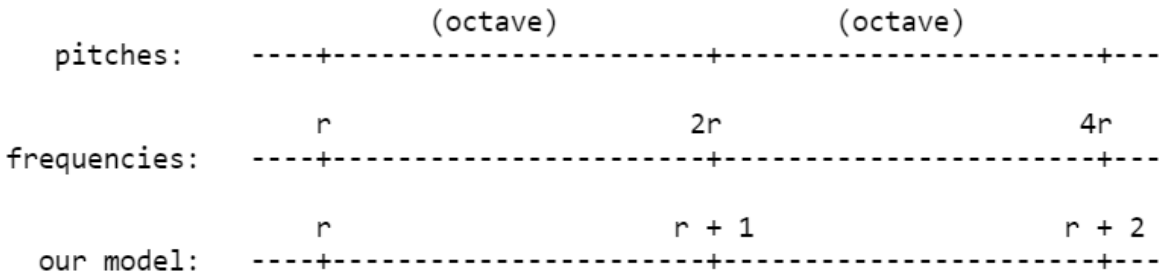
MTO 0.3 Examples: Lindley and Turner-Smith, Mathematical Models of Scales

(Note: audio, video, and other interactive examples are only available online)
http://www.mtosmt.org/issues/mto.93.0.3/mto.93.0.3.lindley_turner-smith.php

Figure 1. A sequence of constructions abstracted from the pitch continuum

- a) a positive-number line for pitch frequencies
- b) a number line for logarithms to base 2 of the frequencies
- c) equivalence classes of points mod 1 on this number-line (flogs)
- d) pitch-class relations between these equivalence classes (also flogs)
- e) two kinds of generators for groups of pitch-class relations:
 equal-division ($1/n$ converted to a flog) or harmonic (see below)
- f) the pairs, (set, Abelian group), which ensue from these generators
- g) equivalence-class neighborhoods around every point-mod-1 in this set
- h) "ideal systems", whereby every system comprises a pair:
 a finite set of non-overlapping neighborhoods, operated upon by
 a subset of one of our groups (that is, by a "halfgroup")
- i) unlimited scales (repeating in every octave indefinitely)
- j) scales with a highest and lowest note

Figure 2. Three pitches in one pitch class



GIF 1. How to reckon in flogs rather than logs to base 2

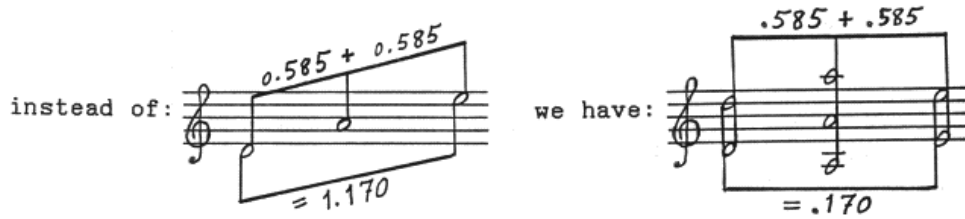
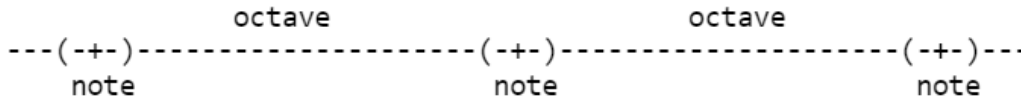


Figure 3. Three notes in one pitch class



GIF 2. Some types of algebraic structure

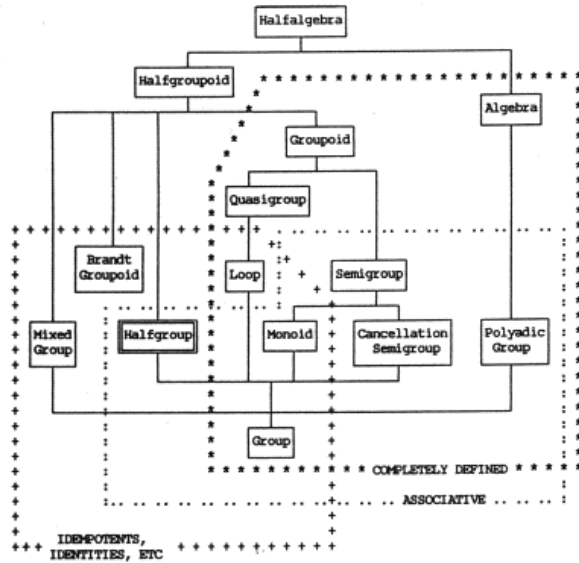


Figure 4. Our notation for the generating harmonic pitch-class relations

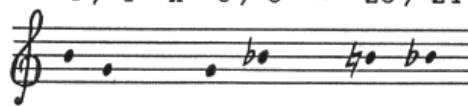
- "I" = flog 2 (= 0, thus the identity element)
- "V" = flog 3 plus or minus a much smaller flog ($t^{\wedge V}$)
- "III" = flog 5 plus or minus a much smaller flog ($t^{\wedge III}$)
- "VII" = flog 7 plus or minus a much smaller flog ($t^{\wedge VII}$)

Figure 5. A rough classification of orders of intervallic magnitude

- (ca. 10 octaves - range of hearing)
- (ca. 1 octave - difference between men's and women's voices)
- 10ths of an octave - melodic steps and leaps
- 100ths of an octave - out-of-tune-ness (Such a t would mar a consonance.)
- 1000ths of an octave - tempering
- 1/10 000-octave - musically insignificant

GIF 3. Reckoning the size of a chromatic semitone

$5/4 \times 5/6 = 25/24$



$\log(25/24) / \log 2 = 0.06$

Figure 6. A table showing how to find the most feasible equations $nV = III$

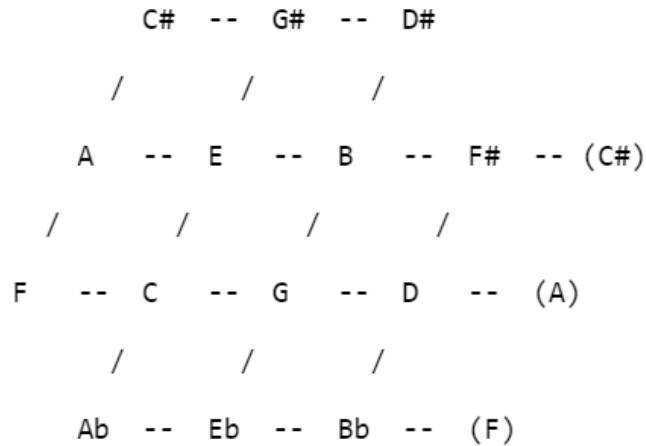
n	$m(\text{flog } 3)$	$\text{flog } 5 = s^{n^{\wedge}}$	$m^{n^{\wedge}}$	$T^{n^{\wedge}} = s^{n^{\wedge}} / (m^{n^{\wedge}} + 1)$	
1	$-\text{flog } 3$	$-\text{flog } 5 = .0931$	1	.0466	
2	$3 \text{ flog } 3$	$+\text{flog } 5 = .0768$	3	.0192	
3	$4 \text{ flog } 3$	$-\text{flog } 5 = .0179$	4	.0036	--> $4V = III$
4	$8 \text{ flog } 3$	$+\text{flog } 5 = .0016$	8	.0002	--> $-8V = III$
5	$45 \text{ flog } 3$	$-\text{flog } 5 = .0014$	45	.00003	

Figure 7. Branches in our system-tree

- a) Among systems:
 - harmonic
 - equal-division
- b) Among harmonic systems:
 - 1-dimensional (the generators are {I, V})
 - 2-dimensional (the generators are {I, V, III})
 - 3-dimensional (the generators are {I, V, III, VII})
- c) Also among harmonic systems:
 - coherent (comprising one chain of V's)
 - not coherent
- d) Among coherent systems:
 - untempered (all $t = 0$)
 - tempered
- e) Among temperaments:
 - regular (all $t^{V^{\wedge}}$ equal, all $t^{III^{\wedge}}$ equal, etc.)
 - semi-regular (all $t^{V^{\wedge}}$ equal, but not all $t^{III^{\wedge}}$ or $t^{VII^{\wedge}}$)
 - irregular ($t^{V^{\wedge}}$ and hence $t^{III^{\wedge}}$ & $t^{VII^{\wedge}}$ varying)
- f) Among regular, two- (and sometimes three-) dimensional temperaments:
 - MT (wherein $4V = III$)
 - QP (wherein $-8V = III$)
- g) Conjunctions of MT and QP:
 - $ET^{1^{\wedge}}$ ($12V = I$; there is no III and no VII)
 - $ET^{2^{\wedge}}$ ($12V = I$; $4V = III = -8V$; there is no VII)
 - $ET^{3^{\wedge}}$ ($12V = I$; $4V = III = -8V$; $-2V = VII = 10V$)
- h) Apart from ET, some musically good possibilities for MT or for QP:
 - meantone temperaments
 - quasi-Pythagorean temperaments
- i) Physically equivalent equal-division systems,
 - for instance, the mere division of the octave into 12 equal parts
 - (a system which is physically equivalent to an ET but has no diatonic or chromatic semitones)
- j) Certain families of such equal-division systems:
 - $F^{1^{\wedge}}$ (characterized by equivalence to MT systems)
 - $F^{2^{\wedge}}$ (characterized by equivalence to QP systems)
- k) A kind of temperament, CT, which includes ET and some irregular temperaments that approximate to an ET within a certain "margin of equivalence" and meet certain other specifications
- l) Some historically important kinds of CT:
 - JSB
 - *temperament ordinaire* (with an accent over the second "e")
- m) A "semi-regular" temperament that is physically equivalent to a quasi-Pythagorean temperament but has two different values for III

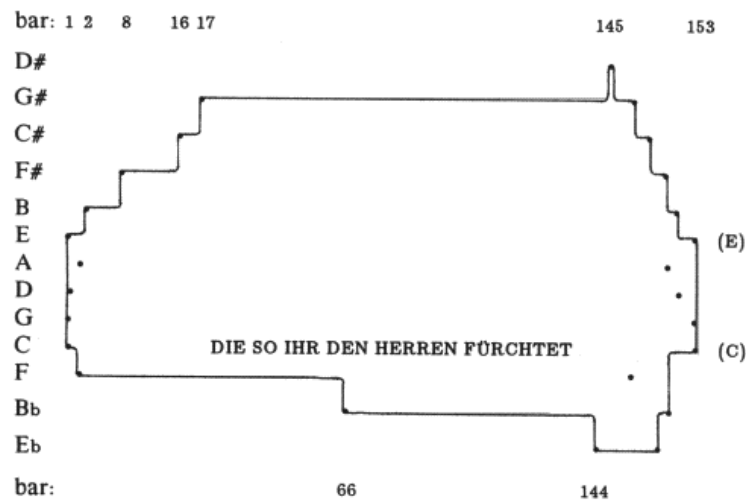
Figure 8. Diagram of a meantone system with 14 pitch classes

("--" between two note-names means there is a V relation between the two pitch classes; "/" means there is a III relation between them. Imagine the "--"s spiraling up on a cylinder from Ab to D#)



GIF 4. Schütz, "Die so ihr den Herren fürchtet" (SWV 164)

(a) A diagram showing when the first and last use of each pitch class occurs



(b-d) Three excerpts showing some of these uses

b. Die so ihr den Her-ren fürch-tet, ver-trau-et ihm, denn es wird euch nicht

1

4 3

feh-len, denn es wird euch nicht feh-len;

4

die so ihr den Her-ren fürch-tet,

c. die so ihr den Her-ren fürch-tet, ver-trau-et,

14

5 6 # 7 6 4 4 #

hof-fet das Be-ste von ihm

d.

143

und hilft und hilft ...

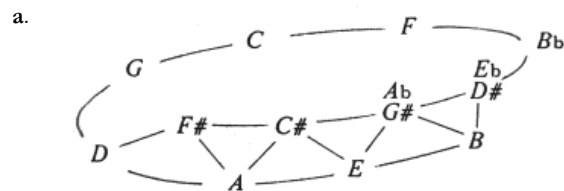
und hilft und hilft ...

148

in der in der Not.

in der Not.

GIF 5. A 15th-century quasi-Pythagorean system and some evidence of its use: music in which triads with an explicit sharp are especially salient



b.

Landini, "O fanciulla giulia" (ballata),
first ending of the second section

Musical score for Landini's "O fanciulla giulia" (ballata), first ending of the second section. The score is in 6/8 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. A first ending bracket is placed over the final two measures of the treble staff, with a first ending fermata symbol above it.

c.

Matteo da Perugia, "A qui fortune" (rondeau),
end of the first section

Musical score for Matteo da Perugia's "A qui fortune" (rondeau), end of the first section. The score is in 7/4 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. A fermata is placed over the final measure of the treble staff.

d.

Matteo da Perugia, "Le grant desir" (ballade),
end of the first section

Musical score for Matteo da Perugia's "Le grant desir" (ballade), end of the first section. The score is in 6/8 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. A fermata is placed over the final measure of the treble staff.

e.

End of an organ verse from the Faenza Codex

Musical score for the end of an organ verse from the Faenza Codex. The score is in 3/4 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. The treble staff contains several triplet markings (indicated by a '3' above the notes) and a fermata over the final measure.

f. Dufay, "Mon chier amy" (ballade),
end of the second section

Musical score for Dufay's "Mon chier amy" (ballade), end of the second section. The score is in 3/4 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. The treble staff contains several triplet markings (indicated by a '3' above the notes) and a fermata over the final measure.

g.

End of a prelude from the Buxheim Organ Book

Musical score for the end of a prelude from the Buxheim Organ Book. The score is in 3/2 time and consists of two staves: a treble staff and a bass staff. The treble staff begins with a treble clef and a key signature of one sharp (F#). The bass staff begins with a bass clef and a key signature of one sharp (F#). The music features a melodic line in the treble and a supporting bass line. A fermata is placed over the final measure of the treble staff.

GIF 6. Lewin's analysis of a passage from *Parsifal*

Figure 9. Some formulas distinguishing F^1 and F^2

Only if $n = 12i + 7j$ (where $i > j$) can $1/n$ generate a system in F^{1^1} .

The following possibilities result when $i = 1, 2, 3$ or 4
and $j = 0, 1$ or 2 :

Values for i:	1	2	3	4
Values for j:	0	12	24	(36) (48)
	1	31	43	55
	2		50	(62)

(The numbers in parentheses are multiples of smaller results in the same table and represent equal-division systems which have *not* played a very substantial role in the history of music theory.)

Only if $n = 12i + 7j$ (where $i > j$) can $1/n$ generate a system in F^{2^1} .

The following possibilities result when $i = 1$ or 4
and $j = 0$ or 1 :

Values for i:	1	4
Values for j:	0	12 (48)
	1	53

GIF 7. A diagram made by Isaac Newton

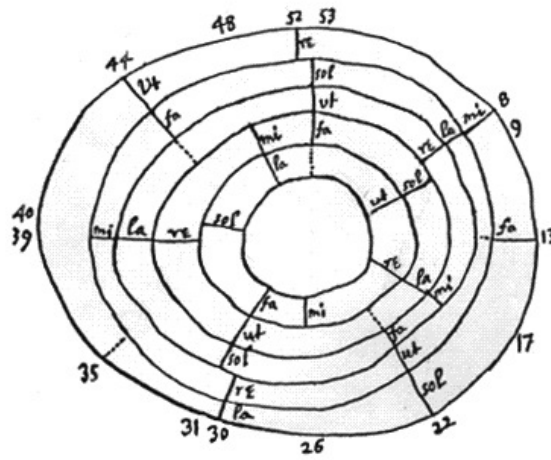
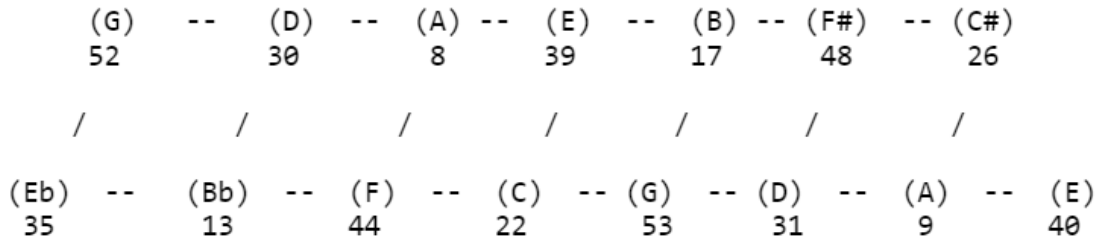
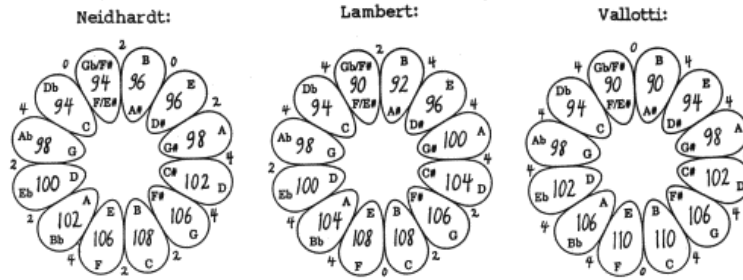


Figure 10. The V and III relations in Newton's system



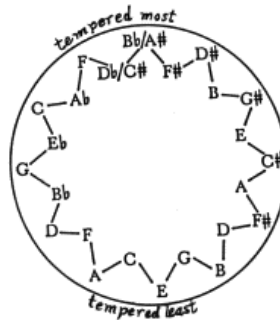
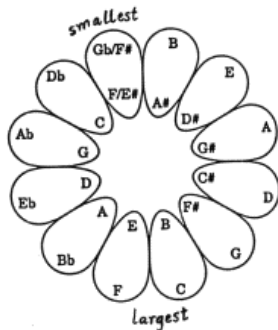
GIF 8. Some 18th-century irregular temperaments



Patterns characteristic of such temperaments

Semitones:

3rds and 6ths:



GIF 9. Louis Couperin, Pavane in F#-minor, first section

A more customary way to resolve the suspension at bar 2:

The image displays five systems of musical notation for the first section of Louis Couperin's Pavane in F#-minor. Each system consists of a treble and bass staff. Asterisks (*) are placed above certain notes in the treble staff of each system, indicating points of interest or suspension. The first system includes a text annotation: "A more customary way to resolve the suspension at bar 2:" followed by a small musical fragment showing a resolution. The systems are numbered 1, 5, 9, 12, and 16 at the beginning of their respective treble staves.

GIF 10. The “margin of equivalence” between two physically different systems with the same number of pitch classes (in this case, four)

